12. CKM Quark-Mixing Matrix

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12.1 Introduction

The masses and mixings of quarks have a common origin in the Standard Model (SM). They arise from the Yukawa interactions with the Higgs condensate,

$$\mathcal{L}_Y = -Y^d_{ij}\, \overline{Q^I_{Li}}\, \phi\, d^I_{Rj} - Y^u_{ij}\, \overline{Q^I_{Li}}\, \epsilon\, \phi^*\, u^I_{Rj} + \text{h.c.}, \tag{12.1}$$

where $Y^{u,d}$ are 3×3 complex matrices, ϕ is the Higgs field, i, j are generation labels, and ϵ is the 2×2 antisymmetric tensor. Q^I_L are left-handed quark doublets, and d^I_R and u^I_R are right-handed down- and up-type quark singlets, respectively, in the weak-eigenstate basis. When ϕ acquires a vacuum expectation value, $\langle\phi\rangle = (0, v/\sqrt{2})$, Eq. (12.1) yields mass terms for the quarks. The physical states are obtained by diagonalizing $Y^{u,d}$ by four unitary matrices, $V^{u,d}_{L,R}$, as $M^f_{\text{diag}} = V^f_L\, Y^f\, V^{f\dagger}_R (v/\sqrt{2})$, $f = u, d$. As a result, the charged-current $W^\pm$ interactions couple to the physical u_{Lj} and d_{Lk} quarks with couplings given by

$$\frac{-g}{\sqrt{2}}(\overline{u_L}, \overline{c_L}, \overline{t_L})\gamma^\mu\, W^+_\mu\, V_{\text{CKM}} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} + \text{h.c.}, \qquad V_{\text{CKM}} \equiv V^u_L V^{d\dagger}_L = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{12.2}$$

This Cabibbo-Kobayashi-Maskawa (CKM) matrix [1, 2] is a 3×3 unitary matrix. It can be parameterized by three mixing angles and the CP-violating KM phase [2]. Of the many possible conventions, a standard choice has become [3]

$$\begin{aligned} V_{\text{CKM}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \end{aligned} \tag{12.3}$$

where $s_{ij} = \sin\theta_{ij}$, $c_{ij} = \cos\theta_{ij}$, and δ is the phase responsible for all CP-violating phenomena in flavor-changing processes in the SM. The angles θ_{ij} can be chosen to lie in the first quadrant, so $s_{ij}, c_{ij} \geq 0$.

It is known experimentally that $s_{13} \ll s_{23} \ll s_{12} \ll 1$, and it is convenient to exhibit this hierarchy using the Wolfenstein parameterization. We define [4–6]

$$\begin{aligned} s_{12} &= \lambda = \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}}, \qquad s_{23} = A\lambda^2 = \lambda\left|\frac{V_{cb}}{V_{us}}\right|, \\ s_{13}e^{i\delta} &= V^*_{ub} = A\lambda^3(\rho + i\eta) = \frac{A\lambda^3(\bar\rho + i\bar\eta)\sqrt{1-A^2\lambda^4}}{\sqrt{1-\lambda^2}\,[1 - A^2\lambda^4(\bar\rho + i\bar\eta)]}. \end{aligned} \tag{12.4}$$

These relations ensure that $\bar\rho + i\bar\eta = -(V_{ud}V^*_{ub})/(V_{cd}V^*_{cb})$ is phase convention independent, and the CKM matrix written in terms of λ, A, $\bar\rho$, and $\bar\eta$ is unitary to all orders in λ. The definitions of $\bar\rho$, $\bar\eta$ reproduce all approximate results in the literature; *i.e.*, $\bar\rho = \rho(1-\lambda^2/2+\ldots)$ and $\bar\eta = \eta(1-\lambda^2/2+\ldots)$, and one can write V_{CKM} to $\mathcal{O}(\lambda^4)$ either in terms of $\bar\rho, \bar\eta$ or, traditionally,

$$V_{\text{CKM}} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4). \tag{12.5}$$

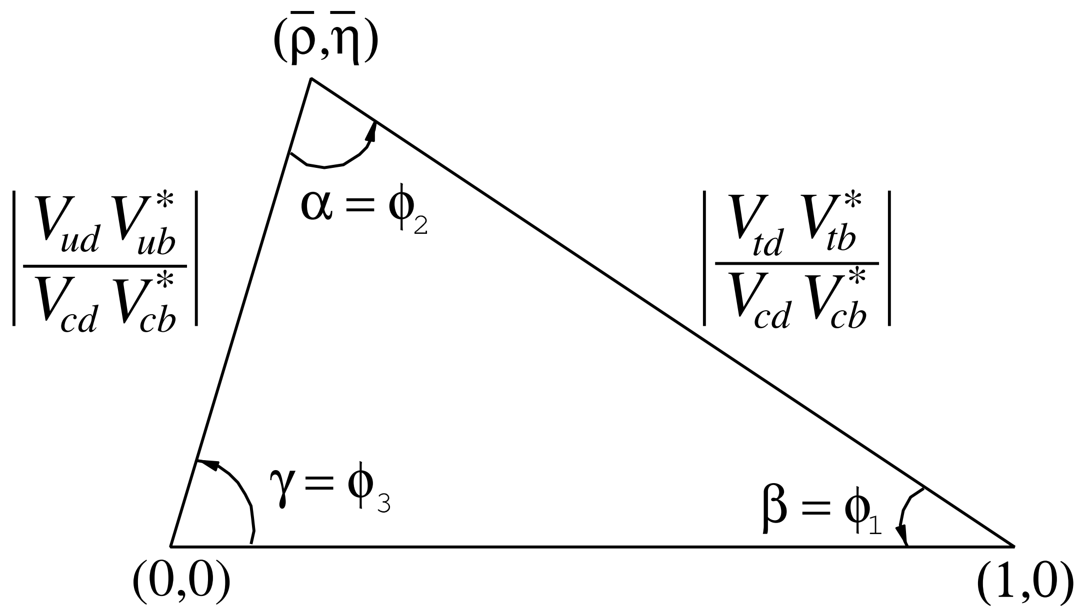

Figure 12.1: Sketch of the unitarity triangle.

The CKM matrix elements are fundamental parameters of the SM, so their precise determination is important. The unitarity of the CKM matrix imposes $\sum_i V_{ij}V_{ik}^* = \delta_{jk}$ and $\sum_j V_{ij}V_{kj}^* = \delta_{ik}$. The six vanishing combinations can be represented as triangles in a complex plane, of which those obtained by taking scalar products of neighboring rows or columns are nearly degenerate. The areas of all triangles are the same, half of the Jarlskog invariant, J [7], which is a phase-convention-independent measure of CP violation, defined by $\mathrm{Im}\big[V_{ij}V_{kl}V_{il}^*V_{kj}^*\big] = J\sum_{m,n}\varepsilon_{ikm}\varepsilon_{jln}$.

The most commonly used unitarity triangle arises from

$$V_{ud}\,V_{ub}^* + V_{cd}\,V_{cb}^* + V_{td}\,V_{tb}^* = 0\,, \tag{12.6}$$

by dividing each side by $V_{cd}V_{cb}^*$ (see Fig. 12.1). Its vertices are exactly $(0,0)$, $(1,0)$, and, due to the definition in Eq. (12.4), $(\bar\rho,\bar\eta)$. An important goal of flavor physics is to overconstrain the CKM elements, and many measurements can be conveniently displayed and compared in the $\bar\rho,\bar\eta$ plane. While the Lagrangian in Eq. (12.1) is renormalized, and the CKM matrix has a well-known scale dependence above the weak scale [8], below $\mu = m_W$ the CKM elements can be treated as constants, with all μ-dependence contained in the running of quark masses and higher-dimension operators.

Unless explicitly stated otherwise, we describe all measurements assuming the SM, to extract magnitudes and phases of CKM elements in Sec. 12.2 and 12.3. Processes dominated by loop-level contributions in the SM are particularly sensitive to new physics beyond the SM (BSM). We give the global fit results for the CKM elements in Sec. 12.4, and discuss some implications for beyond standard model physics in Sec. 12.5.

12.2 Magnitudes of CKM elements

12.2.1 $|V_{ud}|$

The most precise determination of $|V_{ud}|$ comes from the study of superallowed $0^+ \to 0^+$ nuclear beta decays, which are pure vector transitions. Taking the average of the fifteen most precise determinations [9] yields [10]

$$|V_{ud}| = 0.97367 \pm 0.00032\,. \tag{12.7}$$

This uncertainty is slightly more than twice as large as that in the 2020 edition, due to a more conservative estimate of the nuclear structure uncertainties. A less precise determination of $|V_{ud}|$ can be obtained from the measurement of the neutron lifetime. The theoretical uncertainties are very small, but the determination is limited by the knowledge of the ratio of the axial-vector and vector couplings, $g_A = G_A/G_V$ [10]. The PIBETA experiment [11] has improved the measurement of the $\pi^+ \to \pi^0 e^+ \nu$ branching ratio to 0.6%, and Ref. [12] quotes $|V_{ud}| = 0.9739 \pm 0.0027$, in agreement with the more precise result listed above. The interest in this measurement is that the determination of $|V_{ud}|$ is very clean theoretically, because it is a pure vector transition and is free from nuclear-structure uncertainties.

12.2.2 $|V_{us}|$

The product of $|V_{us}|$ and the form factor at $q^2 = 0$, $|V_{us}|\, f_+(0)$, has been extracted traditionally from $K^0_L \to \pi e \nu$ decays in order to avoid isospin-breaking corrections ($\pi^0 - \eta$ mixing) that affect $K^\pm$ semileptonic decay, and the complications induced by a second (scalar) form factor present in the muonic decays. The last round of measurements has led to enough experimental constraints to justify the comparison between different decay modes. Systematic errors related to the experimental quantities, *e.g.*, the lifetime of neutral or charged kaons, and the form factor determinations for electron and muonic decays, differ among decay modes, and the consistency between different determinations enhances the confidence in the final result. For this reason, we follow the prescription [13] to average $K^0_L \to \pi e \nu$, $K^0_L \to \pi \mu \nu$, $K^\pm \to \pi^0 e^\pm \nu$, $K^\pm \to \pi^0 \mu^\pm \nu$ and $K^0_S \to \pi e \nu$. The average of these five decay modes yields $|V_{us}|\, f_+(0) = 0.21656 \pm 0.00035$. Results obtained from each decay mode, and exhaustive references to the experimental data, are listed for instance in Ref. [10]. The form factor average $f_+(0) = 0.9698 \pm 0.0017$ [14] from $N_f = 2 + 1 + 1$ lattice QCD calculations gives $|V_{us}| = 0.2233 \pm 0.0005$ [10].[1] The broadly used classic calculation of $f_+(0)$ [16] is in good agreement with this value, while other calculations [18] differ by as much as 2%.

The calculation of the ratio of the kaon and pion decay constants enables one to extract $|V_{us}/V_{ud}|$ from $K \to \mu\nu(\gamma)$ and $\pi \to \mu\nu(\gamma)$, where (γ) indicates that radiative decays are included [19]. The value of $\Gamma(K \to \mu\nu(\gamma))$ [10] derived from the KLOE measurement of the corresponding branching ratio [20], combined with the lattice QCD result, $f_K/f_\pi = 1.1932 \pm 0.0021$ [14], leads to $|V_{us}| = 0.2250 \pm 0.0004$, where the accuracy is limited by the knowledge of the ratio of the decay constants. The average of these two determinations, with the error scaled according to the PDG prescription [21] by $\sqrt{\chi^2} = 2.5$, is quoted as [10]

$$|V_{us}| = 0.22431 \pm 0.00085\,. \tag{12.8}$$

It is important to include both QED and QCD sources of isospin violations in the lattice QCD calculations.

The latest determination from hyperon decays can be found in Ref. [22]. The authors focus on the analysis of the vector form factor, protected from first order flavor $SU(3)$ breaking effects by the Ademollo-Gatto theorem [23], and treat the ratio between the axial and vector form factors g_1/f_1 as experimental input, thus avoiding first order $SU(3)$ breaking effects in the axial-vector contribution. They find $|V_{us}| = 0.2250 \pm 0.0027$, although this does not include an estimate of the theoretical uncertainty due to second-order $SU(3)$ breaking, contrary to Eq. (12.8). Concerning hadronic τ decays to strange particles, averaging the inclusive decay and the exclusive $\tau \to h\nu$ ($h = \pi$, K) measurements yields $|V_{us}| = 0.2207 \pm 0.0014$ [24].

12.2.3 $|V_{cd}|$

The magnitude of V_{cd} can be extracted from semileptonic charm decays, using theoretical knowledge of the form factors. In semileptonic D decays, lattice QCD calculations have predicted the nor-

[1] For lattice QCD inputs, we use the averages from Ref. [14], unless the minireviews [10,15] choose different values. We only use unquenched results, and if both $N_f = 2 + 1 + 1$ and $2 + 1$ calculations are available, we use the former.

malization of the $D \to \pi\ell\nu$ and $D \to K\ell\nu$ form factors [14]. The dependence on the invariant mass of the lepton pair, q^2, is determined from lattice QCD and theoretical constraints from analyticity [15]. Using $N_f = 2{+}1{+}1$ lattice QCD calculations for $D \to \pi\ell\nu$, $f_+^{D\pi}(0) = 0.612{\pm}0.035$ [14], and the average [24] of the measurements of $D \to \pi\ell\nu$ decays by *BABAR* [25], BESIII [26], CLEO-c [27], and Belle [28], one obtains $|V_{cd}| = 0.2330 \pm 0.0029 \pm 0.0133$, where the first uncertainty is experimental, and the second is from the theoretical uncertainty of the form factor.

The determination of $|V_{cd}|$ is also possible from the leptonic decay $D^+ \to \mu^+\nu$ and $\tau^+\nu$. The experimental uncertainties have not decreased significantly recently. Averaging the BESIII [29] and earlier CLEO [30] for $\mu^+\nu$ and BESIII [31] for $\tau^+\nu$ measurements, and using the $N_f = 2+1+1$ lattice QCD result, $f_D = 212.0 \pm 0.7$ MeV [14], yields $|V_{cd}| = 0.2181 \pm 0.0049 \pm 0.0007$ [24] [2]

Earlier determinations of $|V_{cd}|$ came from neutrino scattering data. The difference of the ratio of double-muon to single-muon production by neutrino and antineutrino beams is proportional to the charm cross section off valence d quarks, and therefore to $|V_{cd}|^2$ times the average semileptonic branching ratio of charm mesons, $\mathcal{B}_\mu$. The method was used first by CDHS [32] and then by CCFR [33] and CHARM II [34]. Averaging these results is complicated, because it requires assumptions about the scale of the QCD corrections, and because $\mathcal{B}_\mu$ is an effective quantity, which depends on the specific neutrino beam characteristics. With no recent experimental input available, we quote the average from a past review, $\mathcal{B}_\mu|V_{cd}|^2 = (0.463 \pm 0.034) \times 10^{-2}$ [35]. Analysis cuts make these experiments insensitive to neutrino energies smaller than 30 GeV. Thus, $\mathcal{B}_\mu$ should be computed using only neutrino interactions with visible energy larger than 30 GeV. An appraisal [36] based on charm-production fractions measured in neutrino interactions [37] gives $\mathcal{B}_\mu = 0.088 \pm 0.006$. Data from the CHORUS experiment [38] are sufficiently precise to extract $\mathcal{B}_\mu$ directly, by comparing the number of charm decays with a muon to the total number of charmed hadrons found in the nuclear emulsions. Requiring the visible energy to be larger than 30 GeV, CHORUS found $\mathcal{B}_\mu = 0.085 \pm 0.009 \pm 0.006$. We use the average of these two determinations, $\mathcal{B}_\mu = 0.087 \pm 0.005$, and obtain $|V_{cd}| = 0.230 \pm 0.011$. Averaging the three determinations above, we find

$$|V_{cd}| = 0.221 \pm 0.004\,. \tag{12.9}$$

12.2.4 $|V_{cs}|$

The direct determination of $|V_{cs}|$ is possible from semileptonic D or leptonic D_s decays, using lattice QCD calculations of the semileptonic D form factor or the D_s decay constant. For muonic decays, the average of Belle [39], CLEO-c [40], *BABAR* [41], and BESIII [42,43] is $\mathcal{B}(D_s^+ \to \mu^+\nu) = (5.43 \pm 0.16) \times 10^{-3}$ [24]. For decays to τ leptons, the average of CLEO-c [40,44], *BABAR* [41], Belle [39], and BESIII [42] gives $\mathcal{B}(D_s^+ \to \tau^+\nu) = (5.40{\pm}0.23){\times}10^{-2}$ [24]. From each of these values, determinations of $|V_{cs}|$ can be obtained using the PDG values for the mass and lifetime of the D_s, the masses of the leptons, and $f_{D_s} = (249.9 \pm 0.5)$ MeV [14]. The average of these determinations gives $|V_{cs}| = 0.984 \pm 0.012$, where the error is dominated by the experimental uncertainty. In semileptonic D decays, lattice QCD calculations of the $D \to K\ell\nu$ form factor are available [14]. Using $f_+^{DK}(0) = 0.7385 \pm 0.0044$ and the average [24] of CLEO-c [27], Belle [28], *BABAR* [45], and recent BESIII [26,46] measurements of $D \to K\ell\nu$ decays, one obtains $|V_{cs}| = 0.972 \pm 0.007$, where the dominant uncertainty is from the theoretical calculation of the form factor. Averaging the determinations from leptonic and semileptonic decays, we find

$$|V_{cs}| = 0.975 \pm 0.006\,. \tag{12.10}$$

Measurements of on-shell $W^\pm$ decays sensitive to $|V_{cs}|$ were made by LEP-2. The W branching ratios depend on the six CKM elements involving only the quarks lighter than m_W. The W

[2] Hereafter the first error is statistical and the second is systematic, unless mentioned otherwise.

branching ratio to each lepton flavor is $1/\mathcal{B}(W \to \ell\bar{\nu}_\ell) = 3\big[1+\sum_{u,c,d,s,b} |V_{ij}|^2\,(1+\alpha_s(m_W)/\pi)+\ldots\big]$. Assuming lepton universality, the measurement $\mathcal{B}(W \to \ell\bar{\nu}_\ell) = (10.83 \pm 0.07 \pm 0.07)\,\%$ [47] implies $\sum_{u,c,d,s,b} |V_{ij}|^2 = 2.002 \pm 0.027$. This is a precise test of unitarity; however, only flavor-tagged W-decays determine $|V_{cs}|$ directly, such as DELPHI's tagged $W^+ \to c\bar{s}$ analysis, yielding $|V_{cs}| = 0.94^{+0.32}_{-0.26} \pm 0.13$ [48].

12.2.5 $|V_{cb}|$

This matrix element can be determined from exclusive and inclusive semileptonic decays of B mesons to charm. The inclusive determinations use the semileptonic decay rate measurement, together with (certain moments of) the lepton energy and the hadronic invariant-mass spectra. The theoretical basis is the operator product expansion [49, 50], which allows calculation of the decay rate and various spectra as expansions in α_s and inverse powers of the heavy-quark mass. The dependence on m_b, m_c, and the parameters that occur at subleading order is different for different moments. The measurements of many moments overconstrain these parameters, and also test the consistency of their determination. The precise extraction of $|V_{cb}|$ requires using a "threshold" quark mass definition [51, 52]. Inclusive measurements have been performed using B mesons from Z^0 decays at LEP, and at e^+e^- colliders operated at the $\Upsilon(4S)$. At LEP, the large boost of B mesons from the Z^0 decay allows the determination of the moments throughout phase space, which is not possible otherwise, but the large statistics available at the B factories lead to more precise determinations. An average of the measurements and a compilation of the references are provided in Ref. [15]: $|V_{cb}| = (42.2 \pm 0.5) \times 10^{-3}$.

Complementary determinations are based on exclusive semileptonic B decays to D and D^*. In the $m_{b,c} \gg \Lambda_{\rm QCD}$ limit, all form factors are given by a single Isgur-Wise function [53], which depends on the product of the four-velocities of the B and $D^{(*)}$ mesons, $w = v \cdot v'$. Heavy-quark symmetry determines the rate (in the symmetry limit) at $w = 1$, the point in phase space at which the invariant mass of the $\ell\bar{\nu}$ pair is maximal; $|V_{cb}|$ is obtained from extrapolating a fit to the spectrum to $w = 1$. The current update of the V_{cb} and V_{ub} minireview quotes from exclusive decays $|V_{cb}| = (39.8 \pm 0.6) \times 10^{-3}$ [15], based on the only unfolded measurement of $B \to D^*$ semileptonic decay distributions [54], and using a more general fit [55] than in earlier B factory measurements. With the uncertainty scaled by $\sqrt{\chi^2} = 3.0$, this yields the combination [15],

$$|V_{cb}| = (41.1 \pm 1.2) \times 10^{-3}\,. \tag{12.11}$$

Determinations of $|V_{cb}|$ that are currently less precise and not included in this average, can be obtained from the measurement of $B_s \to D_s^{(*)}\mu\bar{\nu}$ decays [56]. In addition, semileptonic decays to τ leptons measured in $B \to D^{(*)}\tau\bar{\nu}$ and related modes are also sensitive to $|V_{cb}|$. The most precise data involving τ leptons are the $|V_{cb}|$-independent ratios, $\mathcal{B}(B \to D^{(*)}\tau\bar{\nu})/\mathcal{B}(B \to D^{(*)}\ell\bar{\nu})$ measured by BaBar, Belle, and LHCb. If the current, approximately $3.3\,\sigma$ [24] hint of lepton non-universality prevails, the determination of $|V_{cb}|$ becomes more complicated.

12.2.6 $|V_{ub}|$

The determination of $|V_{ub}|$ from inclusive $B \to X_u\ell\bar{\nu}$ decay is complicated due to large $B \to X_c\ell\bar{\nu}$ backgrounds. In most regions of phase space where the charm background is kinematically forbidden, the hadronic physics enters via unknown nonperturbative functions, so-called shape functions. (By contrast, the nonperturbative physics for $|V_{cb}|$ is encoded in a few parameters.) At leading order in $\Lambda_{\rm QCD}/m_b$, there is only one shape function, which can be extracted from the photon energy spectrum in $B \to X_s\gamma$ [57, 58], and applied to several spectra in $B \to X_u\ell\bar{\nu}$. The subleading shape functions are modeled in the current determinations. Phase space cuts for which the rate has only subleading dependence on the shape function are also possible [59]. The measurements of both the hadronic and the leptonic systems are important for an optimal choice of phase space. A different approach is to make the measurements more inclusive by extending them deeper into the $B \to X_c \ell\bar{\nu}$ region, and thus reduce the theoretical uncertainties. Analyses of the electron-energy endpoint from CLEO [60], *BaBar* [61], and Belle [62] quote $B \to X_u e\bar{\nu}$ partial rates for $|\vec{p}_e| \geq 2.0\,\text{GeV}$ and $1.9\,\text{GeV}$, which are well below the charm endpoint. The large and pure $B\overline{B}$ samples at the B factories permit the selection of $B \to X_u \ell\bar{\nu}$ decays in events where the other B is fully reconstructed [63]. With this full-reconstruction tag method, the four-momenta of both the leptonic and the hadronic final states can be measured. It also gives access to a wider kinematic region, because of improved signal purity. Ref. [15] quotes the inclusive average, $|V_{ub}| = (4.13 \pm 0.12\ ^{+0.13}_{-0.14} \pm 0.18) \times 10^{-3}$, where the first error is experimental, the second arises from the model dependence quoted by the individual measurements, and the third is an additional one estimated in Ref. [15].

To extract $|V_{ub}|$ from exclusive decays, the form factors have to be known. Experimentally, better signal-to-background ratios are offset by smaller yields. The $B \to \pi\ell\bar{\nu}$ branching ratio is now known to 5%. Lattice QCD calculations of the $B \to \pi\ell\bar{\nu}$ form factor are available [64] for the high q^2 region ($q^2 > 16$ or $18\,\text{GeV}^2$). A fit to the experimental partial rates and lattice QCD results versus q^2 yields $|V_{ub}| = (3.70 \pm 0.10 \pm 0.12) \times 10^{-3}$ [24]. Using additional input from light-cone QCD sum rules (which are thought to be reliable in the small q^2 region), yield a combination, $|V_{ub}| = (3.67 \pm 0.09 \pm 0.12) \times 10^{-3}$ [15, 24].

The uncertainties in extracting $|V_{ub}|$ from inclusive and exclusive decays are different to a large extent. An average of these determinations, with the uncertainty scaled by $\sqrt{\chi^2} = 1.4$, is [15]

$$|V_{ub}| = (3.82 \pm 0.20) \times 10^{-3}\,. \tag{12.12}$$

A determination of $|V_{ub}|$ not included in this average can be obtained from $\mathcal{B}(B \to \tau\bar{\nu}) = (1.09 \pm 0.21) \times 10^{-4}$ [24]. Using $f_B = (190.0 \pm 1.3)\,\text{MeV}$ [14] and $\tau_{B^\pm} = (1.638 \pm 0.004)\,\text{ps}$ [65], we find the remarkably consistent result, $|V_{ub}| = (4.11 \pm 0.39) \times 10^{-3}$. This decay is sensitive, for example, to tree-level charged Higgs contributions, and the measured rate is consistent with the SM expectation. The LHCb measurement $|V_{ub}/V_{cb}| = 0.083 \pm 0.004$ [15] from the ratios of $\Lambda_b \to p^+\mu^-\bar{\nu}$ and $\Lambda_b \to \Lambda_c^+\mu^-\bar{\nu}$ [66] and $B_s^0 \to K^-\mu^+\nu$ and $B_s^0 \to D_s^-\mu^+\nu$ [67] in different regions of q^2, provides another complementary determination.

12.2.7 $|V_{td}|$ *and* $|V_{ts}|$

The CKM elements $|V_{td}|$ and $|V_{ts}|$ are not likely to be precisely measurable in tree-level processes involving top quarks, so one has to rely on determinations from $B^0 - \overline{B}^0$ mixing, dominated by box diagrams with top quarks, or loop-mediated rare K and B decays. Theoretical uncertainties in hadronic effects limit the accuracy of the current determinations. These can be reduced by taking ratios of processes that are equal in the flavor $SU(3)$ limit to determine $|V_{td}/V_{ts}|$.

The phenomenon of $B^0 - \overline{B}^0$ mixing was discovered by ARGUS [68], and the mass difference is now precisely measured as $\Delta m_d = (0.5069 \pm 0.0019)\,\text{ps}^{-1}$ [69]. In the B_s^0 system, Δm_s was first measured significantly by CDF [70] and the world average, dominated by an LHCb measurement [71], is $\Delta m_s = (17.765 \pm 0.006)\,\text{ps}^{-1}$ [69]. Neglecting corrections suppressed by $|V_{tb}| - 1$, and using the lattice QCD results $f_{B_d}\sqrt{\widehat{B}_{B_d}} = (210.6 \pm 5.5)\,\text{MeV}$ and $f_{B_s}\sqrt{\widehat{B}_{B_s}} = (256.1 \pm 5.7)\,\text{MeV}$ [14],

$$|V_{td}| = (8.6 \pm 0.2) \times 10^{-3}\,, \qquad |V_{ts}| = (41.5 \pm 0.9) \times 10^{-3}\,. \tag{12.13}$$

The uncertainties are dominated by lattice QCD. Several uncertainties are reduced in the calculation of the ratio $\xi = (f_{B_s}\sqrt{\widehat{B}_{B_s}})/(f_{B_d}\sqrt{\widehat{B}_{B_d}}) = 1.216 \pm 0.016$ [14] and therefore the constraint on $|V_{td}/V_{ts}|$ from $\Delta m_d/\Delta m_s$ is more reliable theoretically. These provide a theoretically clean and significantly improved determination,

$$|V_{td}/V_{ts}| = 0.207 \pm 0.001 \pm 0.003\,. \tag{12.14}$$

The inclusive branching ratio $\mathcal{B}(B \to X_s\gamma) = (3.49 \pm 0.19) \times 10^{-4}$ extrapolated to $E_\gamma > E_0 = 1.6\,\text{GeV}$ [24] is also sensitive to $|V_{tb}V_{ts}|$. In addition to t-quark penguins, a substantial part of the rate comes from charm contributions proportional to $V_{cb}V_{cs}^*$ via the application of 3×3 CKM unitarity (which is used here). With the NNLO calculation of $\mathcal{B}(B \to X_s\gamma)_{E_\gamma>E_0}/\mathcal{B}(B \to X_ce\bar{\nu})$ [72], we obtain $|V_{ts}/V_{cb}| = 1.00 \pm 0.04$. The $B_s \to \mu^+\mu^-$ rate is also proportional to $|V_{tb}V_{ts}|^2$ in the SM, and the world average, $\mathcal{B}(B_s \to \mu^+\mu^-) = (3.01 \pm 0.35) \times 10^{-9}$ [65], is consistent with the SM, with sizable uncertainties.

A complementary determination of $|V_{td}/V_{ts}|$ is possible from the ratio of $B \to \rho\gamma$ and $K^*\gamma$ rates. The ratio of the neutral modes is theoretically cleaner than that of the charged ones, because the poorly known spectator-interaction contribution is expected to be smaller (W-exchange vs. weak annihilation). For now, because of low statistics, we average the charged and neutral rates assuming the isospin symmetry and heavy-quark limit motivated relation, $|V_{td}/V_{ts}|^2/\xi_\gamma^2 = [\mathit{\Gamma}(B^+ \to \rho^+\gamma) + 2\mathit{\Gamma}(B^0 \to \rho^0\gamma)]/[\mathit{\Gamma}(B^+ \to K^{*+}\gamma) + \mathit{\Gamma}(B^0 \to K^{*0}\gamma)] = (3.35 \pm 0.48)\%$ [24,73]. Here ξ_γ contains the poorly known hadronic physics. Using $\xi_\gamma = 1.2 \pm 0.2$ [74] gives $|V_{td}/V_{ts}| = 0.220 \pm 0.016 \pm 0.037$, where the first uncertainty is experimental and the second is theoretical.

A theoretically clean determination of $|V_{td}V_{ts}^*|$ is possible from $K^+ \to \pi^+\nu\bar{\nu}$ decay [75]. Experimentally, more than 20 candidates have been observed [76,77] and the rate is consistent with the SM within errors. Much more data are needed for a precision measurement.

12.2.8 $|V_{tb}|$

The determination of $|V_{tb}|$ from top decays uses the ratio of branching fractions $R = \mathcal{B}(t \to Wb)/\mathcal{B}(t \to Wq) = |V_{tb}|^2/(\sum_q |V_{tq}|^2) = |V_{tb}|^2$, where $q = b, s, d$. The CDF and DØ measurements performed on data collected during Run II of the Tevatron give $|V_{tb}| > 0.85$ [78] and $0.99 > |V_{tb}| > 0.90$ [79], respectively, at 95% CL. CMS measured the same quantity at 8 TeV and obtained $|V_{tb}| > 0.975$ [80] at 95% CL.

The direct determination of $|V_{tb}|$, without assuming unitarity, is possible from the single top quark production cross section. The $(3.30^{+0.52}_{-0.40})$ pb combined cross section [81] of DØ and CDF measurements implies $|V_{tb}| = 1.02^{+0.06}_{-0.05}$. The LHC experiments, ATLAS and CMS, have measured single top quark production cross sections (and extracted $|V_{tb}|$) in t-channel, Wt-channel, and s-channel at 7 TeV, 8 TeV, and 13 TeV [82]. The average of these $|V_{tb}|$ values is calculated to be $|V_{tb}| = 1.007 \pm 0.030$, where all systematic errors and theoretical errors are treated to be fully correlated. The average of Tevatron and LHC values gives

$$|V_{tb}| = 1.010 \pm 0.027\,. \tag{12.15}$$

The experimental systematic uncertainties dominate, and a dedicated combination would be welcome.

A weak constraint on $|V_{tb}|$ can be obtained from precision electroweak data, where top quarks enter in loops. The sensitivity is best in $\mathit{\Gamma}(Z \to b\bar{b})$ and yields $|V_{tb}| = 0.77^{+0.18}_{-0.24}$ [83].

12.3 Phases of CKM elements

As can be seen from Fig. 12.1, the angles of the unitarity triangle are

$$\beta = \phi_1 = \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right),$$
$$\alpha = \phi_2 = \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right),$$
$$\gamma = \phi_3 = \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right). \tag{12.16}$$

Since CP violation involves phases of CKM elements, many measurements of CP-violating observables can be used to constrain these angles and the $\bar\rho, \bar\eta$ parameters.

12.3.1 ϵ *and* ϵ'

The measurement of CP violation in K^0–$\overline{K}^0$ mixing, $|\epsilon| = (2.228 \pm 0.011) \times 10^{-3}$ [84], provides important information about the CKM matrix. The phase of ϵ is determined by long-distance physics, $\epsilon = \frac{1}{2}\, e^{i\phi_\epsilon} \sin\phi_\epsilon \arg(-M_{12}/\Gamma_{12})$, where $\phi_\epsilon = \arctan|2\Delta m_K/\Delta\Gamma_K| \simeq 43.5°$. The SM prediction can be written as

$$\epsilon = \kappa_\epsilon\, e^{i\phi_\epsilon} \frac{G_F^2 m_W^2 m_K}{12\sqrt{2}\,\pi^2 \Delta m_K}\, f_K^2 \widehat{B}_K \Big\{\eta_{tt} S(x_t)\, \mathrm{Im}[(V_{ts}V_{td}^*)^2] + 2\eta_{ct} S(x_c, x_t)\, \mathrm{Im}(V_{cs}V_{cd}^* V_{ts}V_{td}^*) + \eta_{cc}\, x_c\, \mathrm{Im}[(V_{cs}V_{cd}^*)^2]\Big\}, \tag{12.17}$$

where $\kappa_\epsilon \simeq 0.94 \pm 0.02$ [85] includes the effects of strangeness changing $\Delta s = 1$ operators and additional dependence on $\phi_\epsilon \neq \pi/4$ (see also Ref. [86]). The displayed terms are the short-distance $\Delta s = 2$ contribution to $\mathrm{Im}M_{12}$ in the usual phase convention, S is an Inami-Lim function [87], $x_q = m_q^2/m_W^2$, and η_{ij} are perturbative QCD corrections. The constraint from ϵ in the $\bar\rho, \bar\eta$ plane is bounded by approximate hyperbolas. Lattice QCD determined the bag parameter $\widehat{B}_K = 0.717 \pm 0.024$ [14] and the main uncertainties are from $(V_{ts}V_{td}^*)^2$ (approximately given by that of $|V_{cb}|^4$ or A^4), the η_{ij} coefficients, and estimates of κ_ϵ.

The measurement of $6\,\mathrm{Re}(\epsilon'/\epsilon) = 1 - |\eta_{00}/\eta_{+-}|^2$, where each $\eta_{ij} = \langle\pi^i\pi^j|\mathcal{H}|K_L\rangle / \langle\pi^i\pi^j|\mathcal{H}|K_S\rangle$ violates CP, provides a qualitative test of the CKM mechanism, and strong constraints on many BSM scenarios. Its nonzero value, $\mathrm{Re}(\epsilon'/\epsilon) = (1.67 \pm 0.23) \times 10^{-3}$ [84], demonstrated the existence of direct CP violation, a prediction of the KM ansatz. While $\mathrm{Re}(\epsilon'/\epsilon) \propto \mathrm{Im}(V_{td}V_{ts}^*)$, this quantity cannot easily be used to extract CKM parameters, because cancellations between the electromagnetic and gluonic penguin contributions for large m_t [88] enhance the hadronic uncertainties. Most SM estimates [89] agree with the observed value, indicating that $\bar\eta$ is positive. Progress in lattice QCD [90] may yield a precise SM prediction in the future, and trigger new work on assessing the consistency of the SM with the measured value [91, 92].

12.3.2 β / ϕ_1

12.3.2.1 *Charmonium modes*

CP-violation measurements in B-meson decays provide direct information on the angles of the unitarity triangle, shown in Fig. 12.1. These overconstraining measurements serve to improve the determination of the CKM elements, and to reveal possible effects beyond the SM.

The time-dependent CP asymmetry of neutral B decays to a final state f common to B^0 and $\overline{B}^0$ is given by [93–95]

$$\mathcal{A}_f = \frac{\Gamma(\overline{B}^0(t) \to f) - \Gamma(B^0(t) \to f)}{\Gamma(\overline{B}^0(t) \to f) + \Gamma(B^0(t) \to f)} = S_f \sin(\Delta m_d\, t) - C_f \cos(\Delta m_d\, t)\,, \tag{12.18}$$

where

$$S_f = \frac{2\,\mathrm{Im}\lambda_f}{1+|\lambda_f|^2}\,, \qquad C_f = \frac{1-|\lambda_f|^2}{1+|\lambda_f|^2}\,, \qquad \lambda_f = \frac{q}{p}\frac{\bar{A}_f}{A_f}\,. \tag{12.19}$$

Here, q/p describes B^0–$\overline{B}^0$ mixing and, to a good approximation in the SM, $q/p = V_{tb}^* V_{td}/V_{tb}V_{td}^* = e^{-2i\beta+\mathcal{O}(\lambda^4)}$ in the usual phase convention. A_f ($\bar{A}_f$) is the amplitude of the $B^0 \to f$ ($\overline{B}^0 \to f$) decay. If f is a CP eigenstate, and amplitudes with one CKM phase dominate the decay, then $|A_f| = |\bar{A}_f|$, $C_f = 0$, and $S_f = \sin(\arg\lambda_f) = \eta_f \sin 2\phi$, where η_f is the CP eigenvalue of f and 2ϕ is the phase difference between the $B^0 \to f$ and $B^0 \to \overline{B}^0 \to f$ decay paths. A contribution of another amplitude to the decay with a different CKM phase makes the value of S_f sensitive to relative strong-interaction phases between the decay amplitudes (it also makes $C_f \neq 0$ possible).

The $b \to c\bar{c}s$ decays to CP eigenstates ($B^0 \to$ charmonium $K^0_{S,L}$) give currently the most precise measurements of $S_f = -\eta_f \sin 2\beta$. The $b \to s$ penguin amplitudes have dominantly the same weak phase as the $b \to c\bar{c}s$ tree amplitude. Since only λ^2-suppressed penguin amplitudes introduce a different CP-violating phase, amplitudes with a single weak phase dominate, and we expect $\big||\bar{A}_{\psi K}/A_{\psi K}| - 1\big| < 0.01$. The e^+e^- asymmetric-energy B-factory experiments, BaBar [96] and Belle [97], and LHCb [98] provided precise measurements. The world average, including some other measurements, is [24, 95, 99]

$$\sin 2\beta = 0.709 \pm 0.011\,. \tag{12.20}$$

This measurement has a four-fold ambiguity in β, which can be resolved by a global fit as mentioned in Sec. 12.4. Experimentally, the two-fold ambiguity $\beta \to \pi/2-\beta$ (but not $\beta \to \pi+\beta$) can be resolved by a time-dependent angular analysis of $B^0 \to J/\psi K^{*0}$ [100, 101], or a time-dependent Dalitz plot analysis of $B^0 \to \overline{D}^0 h^0$. The time-dependent Dalitz plot analysis of $B^0 \to \overline{D}^0 h^0$ ($h^0 = \pi^0, \eta, \omega$) with $\overline{D}^0 \to K^0_S \pi^+\pi^-$, jointly performed by Belle and BaBar, excludes the $\pi/2-\beta$ solution with 7.3σ confidence level [102]. These results exclude the negative $\cos 2\beta$ solutions, in agreement with the global CKM fit, which is no longer shown in Fig. 12.2.

The $b \to c\bar{c}d$ mediated transitions, such as $B^0 \to J/\psi\pi^0$ and $B^0 \to D^{(*)+}D^{(*)-}$, also measure approximately $\sin 2\beta$. However, the dominant component of the $b \to d$ penguin amplitude has a different CKM phase ($V_{tb}^* V_{td}$) than the tree amplitude ($V_{cb}^* V_{cd}$), and their magnitudes are of the same order in λ. Therefore, the effect of penguins could be large, resulting in $S_f \neq -\eta_f \sin 2\beta$ and $C_f \neq 0$. Such decay modes have been measured by BaBar, Belle, and LHCb. The world averages [24], $S_{J/\psi\pi^0} = -0.86 \pm 0.14$, $S_{J/\psi\rho^0} = -0.66^{+0.16}_{-0.12}$, $S_{D^+D^-} = -0.84 \pm 0.12$, and $S_{D^{*+}D^{*-}} = -0.71 \pm 0.09$ (where $\eta_f = +1$ for the $J/\psi\pi^0$ and D^+D^- modes, while $J/\psi\rho^0$ and $D^{*+}D^{*-}$ are mixtures of CP even and odd states), are consistent with $\sin 2\beta$ obtained from $B^0 \to$ charmonium K^0 decays, and the C_f's are consistent with zero, although the uncertainties are sizable.

The $b \to c\bar{u}d$ decays $B^0 \to \overline{D}^{0(*)} h^0$, with $\overline{D}^0 \to CP$ eigenstates and $\overline{D}^0 \to K^0_S\pi^+\pi^-$ with Dalitz plot analysis, have no penguin contributions, and provide theoretically clean $\sin 2\beta$ measurements. The average of joint analyses of BaBar and Belle data [102, 103] give $\sin 2\beta = 0.71 \pm 0.09$ [24].

12.3.2.2 *Penguin-dominated modes*

The $b \to s\bar{q}q$ penguin-dominated decays have the same CKM phase as the $b \to c\bar{c}s$ tree level decays, up to corrections suppressed by λ^2, since $V_{tb}^* V_{ts} = -V_{cb}^* V_{cs}[1 + \mathcal{O}(\lambda^2)]$. Therefore, decays such as $B^0 \to \phi K^0$ and $\eta' K^0$ provide $\sin 2\beta$ measurements in the SM. Any BSM contribution to the amplitude with a different weak phase would give rise to $S_f \neq -\eta_f \sin 2\beta$, and possibly $C_f \neq 0$. Therefore, the main interest in these modes is not simply to measure $\sin 2\beta$, but to search for new physics. Measurements of many other decay modes in this category, such as $B \to \pi^0 K^0_S$, $K^0_S K^0_S K^0_S$, *etc.*, have also been performed by BaBar and Belle. The results and their uncertainties are summarized in Fig. 13.3 and Table 13.1 of Ref. [94]. The comparison of CP violation measurements between tree-dominated and penguin-dominated modes in B_s^0 decays provides similar sensitivity to new physics.

12.3.3 α / ϕ_2

Since α is the phase between $V_{tb}^* V_{td}$ and $V_{ub}^* V_{ud}$, only time-dependent CP asymmetries in decay modes dominated by $b \to u\bar{u}d$ transitions can directly measure $\sin 2\alpha$, in contrast to $\sin 2\beta$, where several different quark-level transitions can be used. Since $b \to d$ penguin amplitudes have a different CKM phase than $b \to u\bar{u}d$ tree amplitudes, and their magnitudes are of the same order in λ, the penguin contribution can be sizable, which makes the determination of α complicated. To date, α has been measured in $B \to \pi\pi$, $\rho\pi$ and $\rho\rho$ decay modes.

12.3.3.1 $B \to \pi\pi$

It is well-established from the data that there is a sizable contribution of $b \to d$ penguin amplitudes in $B \to \pi\pi$ decays. Thus, $S_{\pi^+\pi^-}$ in the time-dependent $B^0 \to \pi^+\pi^-$ analysis does not measure $\sin 2\alpha$, but

$$S_{\pi^+\pi^-} = \sqrt{1 - C_{\pi^+\pi^-}^2}\, \sin(2\alpha + 2\Delta\alpha)\,, \tag{12.21}$$

where $2\Delta\alpha$ is the phase difference between $e^{2i\gamma}\bar{A}_{\pi^+\pi^-}$ and $A_{\pi^+\pi^-}$. The value of $\Delta\alpha$, and hence α, can be extracted using the isospin relation among the amplitudes of $B^0 \to \pi^+\pi^-$, $B^0 \to \pi^0\pi^0$, and $B^+ \to \pi^+\pi^0$ decays [104],

$$\frac{1}{\sqrt{2}}\, A_{\pi^+\pi^-} + A_{\pi^0\pi^0} - A_{\pi^+\pi^0} = 0\,, \tag{12.22}$$

and a similar expression for the $\bar{A}_{\pi\pi}$'s. This method utilizes the fact that a pair of pions from $B \to \pi\pi$ decay must be in a zero angular momentum state, and, because of Bose statistics, they must have even isospin. Consequently, $\pi^\pm\pi^0$ is in a pure isospin-2 state, while the penguin amplitudes only contribute to the isospin-0 final state. The latter does not hold for the electroweak penguin amplitudes, but their effect is expected to be small. The isospin analysis uses the world averages of *BaBar*, Belle, and LHCb measurements, $S_{\pi^+\pi^-} = -0.666 \pm 0.029$, $C_{\pi^+\pi^-} = -0.311 \pm 0.030$, the decay widths of all three modes, and the direct CP asymmetry $C_{\pi^0\pi^0} = -0.33 \pm 0.22$ [24]. This analysis leads to 16 mirror solutions for $0 \leq \alpha < 2\pi$. Because of this, and due to the experimental uncertainties, some of these solutions are not well separated [95].

12.3.3.2 $B \to \rho\rho$

The decay $B^0 \to \rho^+\rho^-$ contains two vector mesons in the final state, and so in general is a mixture of CP-even and CP-odd components. At the current level of precision, it simplifies the analysis that the longitudinal polarization fractions in $B^+ \to \rho^+\rho^0$ and $B^0 \to \rho^+\rho^-$ decays were measured to be close to unity [105], which implies that the final states are almost purely CP-even. Furthermore, $\mathcal{B}(B^0 \to \rho^0\rho^0) = (0.96 \pm 0.15) \times 10^{-6}$ is much smaller than $\mathcal{B}(B^0 \to \rho^+\rho^-) = (27.5 \pm 1.7) \times 10^{-6}$ and $\mathcal{B}(B^+ \to \rho^+\rho^0) = (23.8 \pm 1.7) \times 10^{-6}$ [24], which implies that the effect of the penguin contributions is small. The isospin analysis using the world averages, $S_{\rho^+\rho^-} = -0.14 \pm 0.13$ and $C_{\rho^+\rho^-} = 0.00 \pm 0.09$ [24], together with the time-dependent CP asymmetry, $S_{\rho^0\rho^0} = -0.3 \pm 0.7$ and $C_{\rho^0\rho^0} = -0.2 \pm 0.9$ [106], and the above mentioned branching fractions and longitudinal polarization fractions, gives two solutions (with mirror solutions at $3\pi/2 - \alpha$) [95]. A possible small violation of Eq. (12.22) due to the finite width of the ρ [107] is so far neglected.

12.3.3.3 $B \to \rho\pi$

The final state in $B^0 \to \rho^+\pi^-$ decay is not a CP eigenstate, but this decay proceeds via the same quark-level diagrams as $B^0 \to \pi^+\pi^-$, and both B^0 and $\overline{B}^0$ can decay to $\rho^+\pi^-$, while the final state in $B^0 \to \rho^0\pi^0$ is a CP eigenstate. Consequently, mixing-induced CP violation can occur in B^0 and $\overline{B}^0$ decays to $\rho^\pm\pi^\mp$ and $\rho^0\pi^0$. The time-dependent Dalitz plot analysis of $B^0 \to \pi^+\pi^-\pi^0$

decays permits the extraction of α with a single discrete ambiguity, $\alpha \to \alpha + \pi$, since one knows the variation of the strong phases in the interference regions of the $\rho^+\pi^-$, $\rho^-\pi^+$, and $\rho^0\pi^0$ amplitudes in the Dalitz plot [108]. The combination of Belle [109] and BABAR [110] measurements gives only moderate constraints [95].

Combining the $B \to \pi\pi$, $\rho\pi$, and $\rho\rho$ decay modes [24, 95, 99], α is constrained as

$$\alpha = (84.1^{+4.5}_{-3.8})^\circ . \qquad (12.23)$$

Similar results can be found in Refs. [111, 112].

12.3.4 γ / ϕ_3

By virtue of Eq. (12.16), γ does not depend on CKM elements involving the top quark, so it can be measured in tree-level B decays. This is an important distinction from the measurements of α and β, and implies that measurements of γ are unlikely to be affected by physics beyond the SM.

12.3.4.1 $B_{(s)} \to D_{(s)}K^{(*)}$

The interference of $B^- \to D^0K^-$ ($b \to c\bar{u}s$) and $B^- \to \overline{D}{}^0K^-$ ($b \to u\bar{c}s$) transitions can be studied in final states accessible in both D^0 and $\overline{D}{}^0$ decays [93]. In principle, it is possible to extract the B and D decay amplitudes, the relative strong phases, and the weak phase γ from the data [95].

A practical complication is that the precision depends sensitively on the ratio of the interfering amplitudes

$$r_B = \left|A(B^- \to \overline{D}{}^0K^-)/A(B^- \to D^0K^-)\right|, \qquad (12.24)$$

which is around 0.1. The original GLW method [113, 114] considers D decays to CP eigenstates, such as $B^\pm \to D^{(*)}_{CP}(\to \pi^+\pi^-)K^{(*)\pm}$. To alleviate the smallness of r_B and make the interfering amplitudes (which are products of the B and D decay amplitudes) comparable in magnitude, the ADS method [115] considers final states where Cabibbo-allowed $\overline{D}{}^0$ and doubly-Cabibbo-suppressed D^0 decays interfere. Measurements have been made by the B factories, CDF, and LHCb, using both methods [24]. The GLW method currently gives only a loose constraint on γ, while the ADS method provides a moderate constraint.

The BPGGSZ method [116, 117] utilizes the fact that both D^0 and $\overline{D}{}^0$ can have large branching fractions to CP self-conjugate three- and four-body final states, such as $K^0_S\pi^+\pi^-$, and the analysis can be optimized by studying the Dalitz plot dependence of the interferences. The best present determination of γ comes from this method, dominated by 3-body D decay modes. Combining results in 3-body decay modes from Belle [118], BABAR [119], and the most precise LHCb [120] one, $\gamma = (70.0 \pm 4.0)^\circ$ is obtained [95]. The uncertainty is sensitive to the central value of the amplitude ratio r_B (and r^*_B for the D^*K mode), for which Belle found somewhat larger central values than BABAR and LHCb. The same values of $r^{(*)}_B$ enter the ADS analyses, and the data (including 4-body D decays [121]) can be combined to fit for $r^{(*)}_B$ and γ. The effect of $D^0-\overline{D}{}^0$ mixing on γ is either below the present experimental accuracy or can be taken into account in the analysis [122] (even if $D^0-\overline{D}{}^0$ mixing is due to CP-violating new physics [123]).

The amplitude ratio is much larger in the analogous $B^0_s \to D^\pm_s K^\mp$ decays, which allows a model-independent extraction of $\gamma - 2\beta_s$ [124] (here $\beta_s = \arg(-V_{ts}V^*_{tb}/V_{cs}V^*_{cb})$ is related to the phase of B_s mixing). Measurements by LHCb with $B^0_s \to D^\pm_s K^\mp$ [125] and $B^0_s \to D^\pm_s K^\mp\pi^+\pi^-$ [126] give $\gamma = (79^{+19}_{-21})^\circ$ using a constraint on $2\beta_s$ (see Sec. 12.5).

Combining all the above measurements [24, 95, 99], γ is constrained as

$$\gamma = (65.7 \pm 3.0)^\circ . \qquad (12.25)$$

Similar results can be found in Refs. [111, 112].

12.3.4.2 $B^0 \to D^{(*)\pm}\pi^\mp$

The interference of $b \to u$ and $b \to c$ transitions can be studied in $\overline{B}^0 \to D^{(*)+}\pi^-$ ($b \to c\bar{u}d$) and $\overline{B}^0 \to B^0 \to D^{(*)+}\pi^-$ ($\bar{b} \to \bar{u}c\bar{d}$) decays and their CP conjugates, since both B^0 and $\overline{B}^0$ decay to $D^{(*)\pm}\pi^\mp$ (or $D^\pm\rho^\mp$, *etc.*). Since there are only tree and no penguin contributions to these decays, in principle, it is possible to extract from the four time-dependent rates the magnitudes of the two hadronic amplitudes, their relative strong phase, and the weak phase between the two decay paths, which is $2\beta + \gamma$.

A complication is that the ratio of the interfering amplitudes is very small, $r_{D\pi} = A(B^0 \to D^+\pi^-)/A(\overline{B}^0 \to D^+\pi^-) = \mathcal{O}(0.01)$ (and similarly for $r_{D^*\pi}$ and $r_{D\rho}$), and therefore it has not been possible to measure it. To obtain $2\beta + \gamma$, $SU(3)$ flavor symmetry and dynamical assumptions have been used to relate $A(\overline{B}^0 \to D^-\pi^+)$ to $A(\overline{B}^0 \to D_s^-\pi^+)$, so this measurement is not model independent at present. Combining the $D^\pm\pi^\mp$, $D^{*\pm}\pi^\mp$ and $D^\pm\rho^\mp$ measurements [127] gives $\sin(2\beta + \gamma) > 0.68$ at 68% CL [111], consistent with the previously discussed results for β and γ.

12.4 Global fit in the Standard Model

Using the independently measured CKM elements mentioned in the previous sections, the unitarity of the CKM matrix can be checked. We obtain $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9984 \pm 0.0007$ (1st row), $|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1.001 \pm 0.012$ (2nd row), $|V_{ud}|^2 + |V_{cd}|^2 + |V_{td}|^2 = 0.9971 \pm 0.0020$ (1st column), and $|V_{us}|^2 + |V_{cs}|^2 + |V_{ts}|^2 = 1.003 \pm 0.012$ (2nd column), respectively. Due to the recent reduction of the value of $|V_{ud}|$, there is a 2.3σ tension with unitarity in the 1st row, leading also to poor consistency of the SM fit below. The uncertainties in the second row and column are dominated by that of $|V_{cs}|$. For the second row, another check is obtained from the measurement of $\sum_{u,c,d,s,b} |V_{ij}|^2$ in Sec. 12.2.4, minus the sum in the first row above: $|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1.002 \pm 0.027$. These provide strong tests of the unitarity of the CKM matrix. With the significantly improved direct determination of $|V_{tb}|$, the unitarity checks for the third row and column have also become fairly precise, leaving decreasing room for mixing with other states. The sum of the three angles of the unitarity triangle, $\alpha + \beta + \gamma = (172 \pm 5)^\circ$, is also consistent with the SM expectation.

The CKM matrix elements can be most precisely determined using a global fit to all available measurements and imposing the SM constraints (*i.e.*, three generation unitarity). The fit must also use theory predictions for hadronic matrix elements, which sometimes have significant uncertainties. There are several approaches to combining the experimental data. CKMfitter [6,111] and Ref. [128] (which develops [129,130] further) use frequentist statistics, while UTfit [112,131] uses a Bayesian approach. These approaches provide similar results.

The constraints implied by the unitarity of the three generation CKM matrix significantly reduce the allowed range of some of the CKM elements. The fit for the Wolfenstein parameters defined in Eq. (12.4) gives

$$\begin{aligned} \lambda &= 0.22501 \pm 0.00068\,, & A &= 0.826^{+0.016}_{-0.015}\,, \\ \bar{\rho} &= 0.1591 \pm 0.0094\,, & \bar{\eta} &= 0.3523^{+0.0073}_{-0.0071}\,. \end{aligned} \tag{12.26}$$

These values are obtained using the method of Refs. [6,111]. The prescription of Refs. [112,131] gives $\lambda = 0.22497 \pm 0.00070$, $A = 0.839 \pm 0.011$, $\bar{\rho} = 0.1581 \pm 0.0092$, and $\bar{\eta} = 0.3548 \pm 0.0072$ [132]; these results are now very close to one another. The fit results for the magnitudes of all nine CKM elements are

$$|V_{\rm CKM}| = \begin{pmatrix} 0.97435 \pm 0.00016 & 0.22501 \pm 0.00068 & 0.003732^{+0.000090}_{-0.000085} \\ 0.22487 \pm 0.00068 & 0.97349 \pm 0.00016 & 0.04183^{+0.00079}_{-0.00069} \\ 0.00858^{+0.00019}_{-0.00017} & 0.04111^{+0.00077}_{-0.00068} & 0.999118^{+0.000029}_{-0.000034} \end{pmatrix}, \tag{12.27}$$

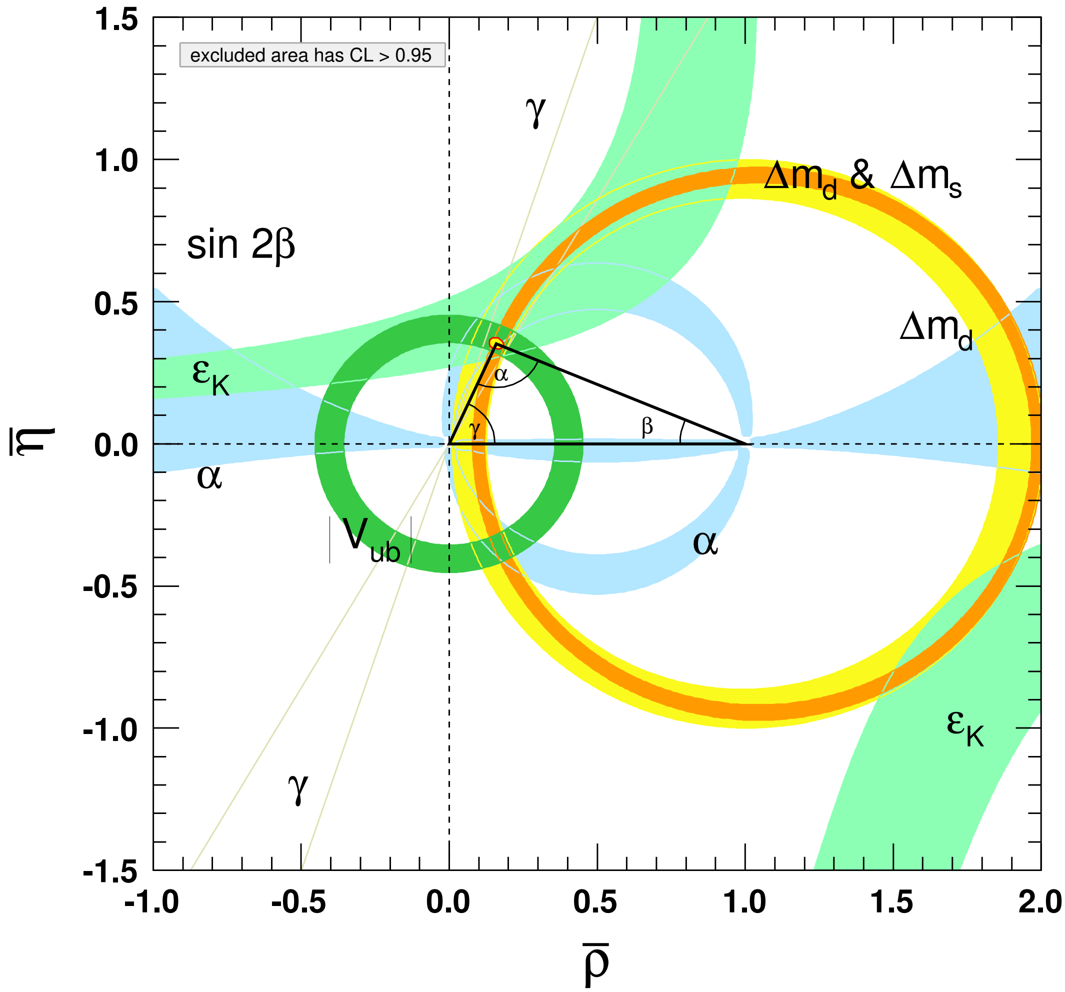

Figure 12.2: Constraints on the $\bar{\rho}, \bar{\eta}$ plane. The shaded areas have 95% CL.

and the Jarlskog invariant is $J = \left(3.12^{+0.13}_{-0.12}\right) \times 10^{-5}$. The parameters in Eq. (12.3) are

$$\begin{aligned} \sin\theta_{12} &= 0.22501 \pm 0.00068\,, & \sin\theta_{13} &= 0.003732^{+0.000090}_{-0.000085}\,, \\ \sin\theta_{23} &= 0.04183^{+0.00079}_{-0.00069}\,, & \delta &= 1.147 \pm 0.026\,. \end{aligned} \tag{12.28}$$

Fig. 12.2 illustrates the constraints on the $\bar{\rho}, \bar{\eta}$ plane from various measurements, and the global fit result. The shaded 95% CL regions all overlap consistently around the global fit region.

If one uses only tree-level inputs (magnitudes of CKM elements not coupling to the top quark and the angle γ), the resulting fit is almost identical for λ in Eq. (12.26), while the other parameters' central values can change by about a sigma and their uncertainties double, yielding $\lambda = 0.22509 \pm 0.00068$, $A = 0.811 \pm 0.024$, $\bar{\rho} = 0.166^{+0.022}_{-0.021}$, and $\bar{\eta} = 0.367^{+0.024}_{-0.023}$. This illustrates how the constraints can be less tight in the presence of BSM physics.

12.5 Implications beyond the SM

The effects in B, B_s, K, and D decays and mixings due to high-scale physics (W, Z, t, H in the SM, and unknown heavier particles) can be parameterized by operators composed of SM fields, obeying the $SU(3) \times SU(2) \times U(1)$ gauge symmetry. Flavor-changing neutral currents, suppressed in the SM, are especially sensitive to beyond SM contributions. Processes studied in great detail, both experimentally and theoretically, include neutral meson mixings, $B_{(s)} \to X\gamma$, $X\ell^+\ell^-$, $\ell^+\ell^-$, $K \to \pi\nu\bar{\nu}$, etc. The BSM contributions to these operators are suppressed by powers of the scale at which they are generated. Already at lowest order, there are many dimension-6 operators, and the observable effects of BSM interactions are encoded in their coefficients. In the SM, these coefficients are determined by just the four CKM parameters, and the W, Z, and quark masses. For example, Δm_d, $\Gamma(B \to \rho\gamma)$, $\Gamma(B \to \pi\ell^+\ell^-)$, and $\Gamma(B \to \ell^+\ell^-)$ are all proportional to $|V_{td}V_{tb}|^2$ in the SM, however, they may receive unrelated BSM contributions. These BSM contributions may or may not obey the SM relations. (For example, the flavor sector of the MSSM contains 69 CP-conserving parameters and 41 CP-violating phases, *i.e.*, 40 new ones [133]). Thus, similar to the measurements of $\sin 2\beta$ in tree- and loop-dominated decay modes, overconstraining measurements of the magnitudes and phases of flavor-changing neutral-current amplitudes gives good sensitivity to BSM.

To illustrate the level of suppression required for BSM contributions, consider a class of models in which the unitarity of the CKM matrix is maintained, and the dominant BSM effects modify the neutral meson mixing amplitudes [134] by $(z_{ij}/\Lambda^2)(\overline{q}_i\gamma^\mu P_L q_j)^2$, where z_{ij} is an unknown coefficient and Λ is the scale suppressing this BSM contribution (see, [135, 136]). It is only known since the first measurements of γ and α that the SM gives the leading contribution to $B^0 - \overline{B}^0$ mixing [6,137]. Nevertheless, new physics with a generic weak phase may still contribute to neutral meson mixings at a significant fraction of the SM [131, 138, 139]. The existing data imply that $\Lambda/|z_{ij}|^{1/2}$ has to exceed about 10^4 TeV for $K^0 - \overline{K}^0$ mixing, 10^3 TeV for $D^0 - \overline{D}^0$ mixing, 500 TeV for $B^0 - \overline{B}^0$ mixing, and 100 TeV for $B_s^0 - \overline{B}_s^0$ mixing [131,136]. (Some other operators are even better constrained [131].) The constraints are the strongest in the kaon sector, because the CKM suppression is the most severe. Thus, if there is new physics at the TeV scale, $|z_{ij}| \ll 1$ is required. Even if $|z_{ij}|$ are suppressed by a loop factor and $|V_{ti}^* V_{tj}|^2$ (in the down quark sector), similar to the SM, one expects percent-level effects, which may be observable in forthcoming flavor physics experiments. To constrain such extensions of the SM, many measurements irrelevant for the SM-CKM fit, such as the CP asymmetry in semileptonic $B_{d,s}^0$ decays, $A_{\rm SL}^{d,s}$, are important [140]. The current world averages [24] are consistent with the SM, with experimental uncertainties far greater than those of the theory predictions.

There are many key measurements sensitive to BSM physics, which do not constrain the unitarity triangle in Fig. 12.1. For example, a key quantity in the B_s system is $\beta_s = \arg(-V_{ts}V_{tb}^*/V_{cs}V_{cb}^*)$, which is the small, λ^2-suppressed, angle of a “squashed” unitarity triangle, obtained by taking the scalar product of the second and third columns of the CKM matrix. This angle can be measured via time-dependent CP violation in $B_s^0 \to J/\psi\,\phi$, similar to β in $B^0 \to J/\psi K^0$. Since the $J/\psi\,\phi$ final state is not a CP eigenstate, an angular analysis of the decay products is needed to separate the CP-even and CP-odd components, which give opposite asymmetries. In the SM, the asymmetry for the CP-even part is $2\beta_s$, when one neglects subdominant amplitudes with a weak phase V_{ub}. (Sometimes the notation $\phi_s = -2\beta_s$ plus a possible BSM contribution to the B_s mixing phase is used.) Testing if the data agree with the SM prediction, $2\beta_s = 0.03726^{+0.00078}_{-0.00077}$ [111], is another sensitive probe of the SM. The current world average, dominated by LHC measurements including the $B_s \to J/\psi\,K^+K^-$ and $J/\psi\,\pi^+\pi^-$ decay modes, is $2\beta_s = 0.040 \pm 0.016$ [69]. Since the uncertainty is much larger than that in the SM, a lot will be learned from more precise future measurements. Searches for CP violation in the charm sector, in particular in $D^0 - \overline{D}^0$ mixing, provide complementary sensitivity to BSM.

In the kaon sector, the CP-violating observables, ϵ and ϵ', are tiny, so models in which all sources of CP violation are small were viable before the B-factory measurements. Since the measurement of $\sin 2\beta$, we know that CP violation can be an $\mathcal{O}(1)$ effect, and only flavor mixing is suppressed between the three quark generations. Thus, many models with spontaneous CP violation were excluded. In the kaon sector, clean tests of the SM can come from measurements of $K^+ \to \pi^+\nu\bar{\nu}$ [77] and $K_L^0 \to \pi^0\nu\bar{\nu}$ [141]. These loop-induced rare decays are sensitive to BSM, and will allow precise tests [142] of the CKM paradigm, independent of B decays.

The CKM elements are fundamental parameters, so they should be measured as precisely as possible. The overconstraining measurements of CP asymmetries, mixing, semileptonic, and rare decays severely constrain the magnitudes and phases of possible BSM contributions to flavor-changing interactions. If new particles are observed at the LHC, it will be important to explore their flavor parameters as precisely as possible to understand the underlying physics.

References

[1] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963).

[2] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[3] L.-L. Chau and W.-Y. Keung, Phys. Rev. Lett. **53**, 1802 (1984).

[4] L. Wolfenstein, Phys. Rev. Lett. **51**, 1945 (1983).

[5] A. J. Buras, M. E. Lautenbacher and G. Ostermaier, Phys. Rev. **D50**, 3433 (1994), [hep-ph/9403384].

[6] J. Charles *et al.* (CKMfitter Group), Eur. Phys. J. **C41**, 1, 1 (2005), [hep-ph/0406184].

[7] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985).

[8] W. J. Marciano and A. Sirlin, Nucl. Phys. **B93**, 303 (1975); K. S. Babu, Z. Phys. **C35**, 69 (1987).

[9] J. C. Hardy and I. S. Towner, Phys. Rev. C **102**, 4, 045501 (2020).

[10] E. Blucher, G. D'Ambrosio, and W.J. Marciano, "V_{ud}, V_{us}, the Cabibbo Angle and CKM Unitarity," in this *Review*.

[11] D. Pocanic *et al.*, Phys. Rev. Lett. **93**, 181803 (2004), [hep-ex/0312030].

[12] A. Czarnecki, W. J. Marciano and A. Sirlin, Phys. Rev. D **101**, 9, 091301 (2020), [arXiv:1911.04685].

[13] M. Antonelli *et al.* (FlaviaNet Working Group on Kaon Decays), Eur. Phys. J. **C69**, 399 (2010), [arXiv:1005.2323]; see also `http://www.lnf.infn.it/wg/vus`.

[14] S. Aoki *et al.* (Flavour Lattice Averaging Group) "FLAG Review 2021 (March 24, 2023 update)", `http://flag.unibe.ch/2021`; The original papers that led to the quoted averages are cited in this reference.

[15] T. Mannel and P. Urquijo, "Semileptonic b-Hadron Decays, Determination of V_{cb} and V_{ub}," in this *Review*.

[16] H. Leutwyler and M. Roos, Z. Phys. **C25**, 91 (1984); For earlier fits for $|V_{ud}|$ and $|V_{us}|$ in the 3-generation SM, see Ref. [17].

[17] R. E. Shrock and L.-L. Wang, Phys. Rev. Lett. **41**, 1692 (1978).

[18] J. Bijnens and P. Talavera, Nucl. Phys. **B669**, 341 (2003), [hep-ph/0303103]; M. Jamin, J. A. Oller and A. Pich, JHEP **02**, 047 (2004), [hep-ph/0401080]; V. Cirigliano *et al.*, JHEP **04**, 006 (2005), [hep-ph/0503108]; C. Dawson *et al.*, PoS **LAT2005**, 337 (2006), [hep-lat/0510018]; N. Tsutsui *et al.* (JLQCD), PoS **LAT2005**, 357 (2006), [hep-lat/0510068]; M. Okamoto (Fermilab Lattice, MILC, HPQCD), in "3rd Conference on Flavor Physics and CP Violation (FPCP 2004) Daegu, Korea, October 4-9, 2004," (2004), [hep-lat/0412044].

[19] W. J. Marciano, Phys. Rev. Lett. **93**, 231803 (2004), [hep-ph/0402299].

[20] F. Ambrosino *et al.* (KLOE), Phys. Lett. **B632**, 76 (2006), [hep-ex/0509045].

[21] See Sec. 5.2, "Averages and fits," in the Introduction to this *Review*, https://pdg.lbl.gov/2023/reviews/rpp2023-rev-rpp-intro.pdf.

[22] N. Cabibbo, E. C. Swallow and R. Winston, Ann. Rev. Nucl. Part. Sci. **53**, 39 (2003), [hep-ph/0307298]; N. Cabibbo, E. C. Swallow and R. Winston, Phys. Rev. Lett. **92**, 251803 (2004), [hep-ph/0307214].

[23] M. Ademollo and R. Gatto, Phys. Rev. Lett. **13**, 264 (1964).

[24] Y. S. Amhis *et al.* (HFLAV), Phys. Rev. D **107**, 5, 052008 (2023), [arXiv:2206.07501]; and updates at https://hflav.web.cern.ch/.

[25] J. P. Lees *et al.* (BaBar), Phys. Rev. **D91**, 5, 052022 (2015), [arXiv:1412.5502].

[26] M. Ablikim *et al.* (BESIII), Phys. Rev. **D92**, 7, 072012 (2015), [arXiv:1508.07560]; M. Ablikim *et al.* (BESIII), Phys. Rev. **D96**, 1, 012002 (2017), [arXiv:1703.09084].

[27] D. Besson *et al.* (CLEO), Phys. Rev. **D80**, 032005 (2009), [arXiv:0906.2983].

[28] L. Widhalm *et al.* (Belle), Phys. Rev. Lett. **97**, 061804 (2006), [hep-ex/0604049].

[29] M. Ablikim *et al.* (BESIII), Phys. Rev. **D89**, 5, 051104 (2014), [arXiv:1312.0374].

[30] B. I. Eisenstein *et al.* (CLEO), Phys. Rev. **D78**, 052003 (2008), [arXiv:0806.2112].

[31] M. Ablikim *et al.* (BESIII), Phys. Rev. Lett. **123**, 21, 211802 (2019), [arXiv:1908.08877].

[32] H. Abramowicz *et al.*, Z. Phys. **C15**, 19 (1982).

[33] S. A. Rabinowitz *et al.*, Phys. Rev. Lett. **70**, 134 (1993); A. O. Bazarko *et al.* (CCFR), Z. Phys. **C65**, 189 (1995), [hep-ex/9406007].

[34] P. Vilain *et al.* (CHARM II), Eur. Phys. J. **C11**, 19 (1999).

[35] F. J. Gilman, K. Kleinknecht and B. Renk (2004).

[36] G. De Lellis, P. Migliozzi and P. Santorelli, Phys. Rept. **399**, 227 (2004), [Erratum: Phys. Rept.411,323(2005)].

[37] N. Ushida *et al.* (Fermilab E531), Phys. Lett. **B206**, 380 (1988); T. Bolton (1997), [hep-ex/9708014].

[38] A. Kayis-Topaksu *et al.* (CHORUS), Phys. Lett. **B626**, 24 (2005).

[39] A. Zupanc *et al.* (Belle), JHEP **09**, 139 (2013), [arXiv:1307.6240].

[40] J. P. Alexander *et al.* (CLEO), Phys. Rev. **D79**, 052001 (2009), [arXiv:0901.1216].

[41] P. del Amo Sanchez *et al.* (BaBar), Phys. Rev. **D82**, 091103 (2010), [Erratum: Phys. Rev.D91,no.1,019901(2015)], [arXiv:1008.4080].

[42] M. Ablikim *et al.* (BESIII), Phys. Rev. **D94**, 7, 072004 (2016), [arXiv:1608.06732].

[43] M. Ablikim *et al.* (BESIII), Phys. Rev. D **104**, 5, 052009 (2021), [arXiv:2102.11734].

[44] P. U. E. Onyisi *et al.* (CLEO), Phys. Rev. **D79**, 052002 (2009), [arXiv:0901.1147]; P. Naik *et al.* (CLEO), Phys. Rev. **D80**, 112004 (2009), [arXiv:0910.3602].

[45] B. Aubert *et al.* (BaBar), Phys. Rev. **D76**, 052005 (2007), [arXiv:0704.0020].

[46] M. Ablikim *et al.* (BESIII), Phys. Rev. Lett. **122**, 1, 011804 (2019), [arXiv:1810.03127].

[47] LEP W branching fraction results for this Review of Particle Physics, LEPEWWG/XSEC/2005-01, http://lepewwg.web.cern.ch/LEPEWWG/lepww/4f/Winter05.

[48] P. Abreu *et al.* (DELPHI), Phys. Lett. **B439**, 209 (1998).

[49] I. I. Y. Bigi *et al.*, Phys. Rev. Lett. **71**, 496 (1993), [hep-ph/9304225].

[50] A. V. Manohar and M. B. Wise, Phys. Rev. **D49**, 1310 (1994), [hep-ph/9308246].

[51] I. I. Y. Bigi *et al.*, Phys. Rev. **D56**, 4017 (1997), [hep-ph/9704245].

[52] A. H. Hoang, Z. Ligeti and A. V. Manohar, Phys. Rev. **D59**, 074017 (1999), [hep-ph/9811239]; A. H. Hoang, Z. Ligeti and A. V. Manohar, Phys. Rev. Lett. **82**, 277 (1999), [hep-ph/9809423]; A. H. Hoang and T. Teubner, Phys. Rev. **D60**, 114027 (1999), [hep-ph/9904468].

[53] N. Isgur and M. B. Wise, Phys. Lett. **B237**, 527 (1990); N. Isgur and M. B. Wise, Phys. Lett. **B232**, 113 (1989).

[54] A. Abdesselam *et al.* (Belle) (2017), [arXiv:1702.01521].

[55] C. G. Boyd, B. Grinstein and R. F. Lebed, Phys. Rev. **D56**, 6895 (1997), [hep-ph/9705252]; C. G. Boyd, B. Grinstein and R. F. Lebed, Nucl. Phys. **B461**, 493 (1996), [hep-ph/9508211].

[56] R. Aaij *et al.* (LHCb), Phys. Rev. D **101**, 7, 072004 (2020), [arXiv:2001.03225].

[57] M. Neubert, Phys. Rev. **D49**, 3392 (1994), [hep-ph/9311325]; M. Neubert, Phys. Rev. **D49**, 4623 (1994), [hep-ph/9312311].

[58] I. I. Y. Bigi *et al.*, Int. J. Mod. Phys. **A9**, 2467 (1994), [hep-ph/9312359].

[59] C. W. Bauer, Z. Ligeti and M. E. Luke, Phys. Lett. **B479**, 395 (2000), [hep-ph/0002161]; C. W. Bauer, Z. Ligeti and M. E. Luke, Phys. Rev. **D64**, 113004 (2001), [hep-ph/0107074].

[60] A. Bornheim *et al.* (CLEO), Phys. Rev. Lett. **88**, 231803 (2002), [hep-ex/0202019].

[61] B. Aubert *et al.* (BaBar), Phys. Rev. **D73**, 012006 (2006), [hep-ex/0509040].

[62] A. Limosani *et al.* (Belle), Phys. Lett. **B621**, 28 (2005), [hep-ex/0504046].

[63] P. Urquijo *et al.* (Belle), Phys. Rev. Lett. **104**, 021801 (2010), [arXiv:0907.0379]; J. P. Lees *et al.* (BaBar), Phys. Rev. **D86**, 032004 (2012), [arXiv:1112.0702].

[64] J. A. Bailey *et al.* (Fermilab Lattice, MILC), Phys. Rev. **D92**, 1, 014024 (2015), [arXiv:1503.07839]; J. M. Flynn *et al.*, Phys. Rev. **D91**, 7, 074510 (2015), [arXiv:1501.05373]; B. Colquhoun *et al.*, Phys. Rev. **D93**, 3, 034502 (2016), [arXiv:1510.07446].

[65] Particle listing, in this *Review*.

[66] R. Aaij *et al.* (LHCb), Nature Phys. **11**, 743 (2015), [arXiv:1504.01568].

[67] R. Aaij *et al.* (LHCb), Phys. Rev. Lett. **126**, 8, 081804 (2021), [arXiv:2012.05143].

[68] H. Albrecht *et al.* (ARGUS), Phys. Lett. **B192**, 245 (1987).

[69] O. Schneider, "B^0–$\overline{B}^0$ mixing," in this *Review*.

[70] A. Abulencia *et al.* (CDF), Phys. Rev. Lett. **97**, 242003 (2006), [hep-ex/0609040].

[71] R. Aaij *et al.* (LHCb), Nature Phys. **18**, 1, 1 (2022), [arXiv:2104.04421].

[72] M. Misiak *et al.*, Phys. Rev. Lett. **114**, 22, 221801 (2015), [arXiv:1503.01789]; P. Czakon, MichaŁEand Fiedler *et al.*, JHEP **04**, 168 (2015), [arXiv:1503.01791].

[73] Heavy Flavor Averaging Group [24] *CP* Asymmetries in Charmless *B* Decay, End of April 2023, `https://hflav-eos.web.cern.ch/hflav-eos/rare/Apr2023/html/index.html`.

[74] B. Grinstein and D. Pirjol, Phys. Rev. **D62**, 093002 (2000), [hep-ph/0002216]; A. Ali, E. Lunghi and A. Ya. Parkhomenko, Phys. Lett. **B595**, 323 (2004), [hep-ph/0405075]; M. Beneke, T. Feldmann and D. Seidel, Nucl. Phys. **B612**, 25 (2001), [hep-ph/0106067]; S. W. Bosch and G. Buchalla, Nucl. Phys. **B621**, 459 (2002), [hep-ph/0106081]; Z. Ligeti and M. B. Wise, Phys. Rev. **D60**, 117506 (1999), [hep-ph/9905277]; D. Becirevic *et al.*, JHEP **05**, 007 (2003), [hep-lat/0301020]; P. Ball, G. W. Jones and R. Zwicky, Phys. Rev. **D75**, 054004 (2007), [hep-ph/0612081]; W. Wang, R.-H. Li and C.-D. Lu (2007), [arXiv:0711.0432]; C.-D. Lu, W. Wang and Z.-T. Wei, Phys. Rev. **D76**, 014013 (2007), [hep-ph/0701265].

[75] A. J. Buras *et al.*, Phys. Rev. Lett. **95**, 261805 (2005), [hep-ph/0508165].

[76] A. V. Artamonov *et al.* (E949), Phys. Rev. Lett. **101**, 191802 (2008), [arXiv:0808.2459]; A. V. Artamonov *et al.* (BNL-E949), Phys. Rev. **D79**, 092004 (2009), [arXiv:0903.0030].

[77] E. Cortina Gil *et al.* (NA62), JHEP **06**, 093 (2021), [arXiv:2103.15389].

[78] T. A. Aaltonen *et al.* (CDF), Phys. Rev. Lett. **112**, 22, 221801 (2014), [arXiv:1404.3392].

[79] V. M. Abazov *et al.* (D0), Phys. Rev. Lett. **107**, 121802 (2011), [arXiv:1106.5436].

[80] V. Khachatryan *et al.* (CMS), Phys. Lett. **B736**, 33 (2014), [arXiv:1404.2292].

[81] T. A. Aaltonen *et al.* (CDF, D0), Phys. Rev. Lett. **115**, 15, 152003 (2015), [arXiv:1503.05027].

[82] LHC Top Working Group summary plots, single top quark production, November 2023, `https://twiki.cern.ch/twiki/bin/view/LHCPhysics/LHCTopWGSummaryPlots`.

[83] J. Swain and L. Taylor, Phys. Rev. **D58**, 093006 (1998), [hep-ph/9712420].

[84] "K_L^0 meson" particle listing, in this *Review*.

[85] A. J. Buras, D. Guadagnoli and G. Isidori, Phys. Lett. **B688**, 309 (2010), [arXiv:1002.3612]; For earlier discussions, see Ref. [86].

[86] E. A. Andriyash, G. G. Ovanesyan and M. I. Vysotsky, Phys. Lett. **B599**, 253 (2004), [hep-ph/0310314]; K. Anikeev *et al.*, in "Workshop on B Physics at the Tevatron: Run II and Beyond Batavia, Illinois, September 23-25, 1999," (2001), [hep-ph/0201071]; A. J. Buras and D. Guadagnoli, Phys. Rev. **D78**, 033005 (2008), [arXiv:0805.3887].

[87] T. Inami and C. S. Lim, Prog. Theor. Phys. **65**, 297 (1981), [Erratum: Prog. Theor. Phys.65,1772(1981)].

[88] J. M. Flynn and L. Randall, Phys. Lett. **B224**, 221 (1989), [Erratum: Phys. Lett.B235,412(1990)]; G. Buchalla, A. J. Buras and M. K. Harlander, Nucl. Phys. **B337**, 313 (1990).

[89] M. Ciuchini *et al.*, Phys. Lett. **B301**, 263 (1993), [hep-ph/9212203]; A. J. Buras, M. Jamin and M. E. Lautenbacher, Nucl. Phys. **B408**, 209 (1993), [hep-ph/9303284]; T. Hambye *et al.*, Nucl. Phys. **B564**, 391 (2000), [hep-ph/9906434]; S. Bertolini, J. O. Eeg and M. Fabbrichesi, Phys. Rev. **D63**, 056009 (2001), [hep-ph/0002234]; V. Cirigliano *et al.*, Phys. Rev. Lett. **91**, 162001 (2003), [hep-ph/0307030].

[90] R. Abbott *et al.* (RBC, UKQCD), Phys. Rev. D **102**, 5, 054509 (2020), [arXiv:2004.09440].

[91] A. J. Buras *et al.*, JHEP **11**, 202 (2015), [arXiv:1507.06345].

[92] V. Cirigliano *et al.*, JHEP **02**, 032 (2020), [arXiv:1911.01359].

[93] A. B. Carter and A. I. Sanda, Phys. Rev. Lett. **45**, 952 (1980); A. B. Carter and A. I. Sanda, Phys. Rev. **D23**, 1567 (1981).

[94] A more detailed discussion and references can be found in: T. Gershon and Y. Nir, "CP violation in meson decays," in this *Review*.

[95] T. Gershon, M. Kenzie and K. Trabelsi, "Determination of CKM angles from B hadrons," in this *Review*.

[96] B. Aubert *et al.* (BaBar), Phys. Rev. **D79**, 072009 (2009), [arXiv:0902.1708].

[97] I. Adachi *et al.* (Belle), Phys. Rev. Lett. **108**, 171802 (2012), [arXiv:1201.4643].

[98] R. Aaij *et al.* (LHCb), Phys. Rev. Lett. **132**, 2, 021801 (2024), [arXiv:2309.09728].

[99] Heavy Flavor Averaging Group [24], Results on Time-Dependent CP Violation and Measurements Related to the Angles of the Unitarity Triangle: Winter conferences (Moriond, etc.) 2024: `https://hflav-eos.web.cern.ch/hflav-eos/triangle/moriond2024/`.

[100] B. Aubert *et al.* (BaBar), Phys. Rev. **D71**, 032005 (2005), [hep-ex/0411016].

[101] R. Itoh *et al.* (Belle), Phys. Rev. Lett. **95**, 091601 (2005), [hep-ex/0504030].

[102] I. Adachi *et al.* (BaBar, Belle), Phys. Rev. Lett. **121**, 26, 261801 (2018), [arXiv:1804.06152]; I. Adachi *et al.* (BaBar, Belle), Phys. Rev. **D98**, 11, 112012 (2018), [arXiv:1804.06153].

[103] A. Abdesselam *et al.* (BaBar, Belle), Phys. Rev. Lett. **115**, 12, 121604 (2015), [arXiv:1505.04147].

[104] M. Gronau and D. London, Phys. Rev. Lett. **65**, 3381 (1990).

[105] J. Zhang *et al.* (Belle), Phys. Rev. Lett. **91**, 221801 (2003), [hep-ex/0306007]; A. Somov *et al.* (Belle), Phys. Rev. Lett. **96**, 171801 (2006), [hep-ex/0601024]; B. Aubert *et al.* (BaBar), Phys. Rev. Lett. **97**, 261801 (2006), [hep-ex/0607092]; B. Aubert *et al.* (BaBar), Phys. Rev. **D76**, 052007 (2007), [arXiv:0705.2157].

[106] B. Aubert *et al.* (BaBar), Phys. Rev. **D78**, 071104 (2008), [arXiv:0807.4977].

[107] A. F. Falk *et al.*, Phys. Rev. **D69**, 011502 (2004), [hep-ph/0310242].

[108] A. E. Snyder and H. R. Quinn, Phys. Rev. **D48**, 2139 (1993).

[109] A. Kusaka *et al.* (Belle), Phys. Rev. Lett. **98**, 221602 (2007), [hep-ex/0701015].

[110] J. P. Lees *et al.* (BaBar), Phys. Rev. **D88**, 1, 012003 (2013), [arXiv:1304.3503].

[111] A. Hocker *et al.*, Eur. Phys. J. **C21**, 225 (2001), [hep-ph/0104062]; and updates at `http://ckmfitter.in2p3.fr/`.

[112] M. Bona *et al.* (UTfit), JHEP **07**, 028 (2005), [hep-ph/0501199]; and updates at `http://www.utfit.org`.

[113] M. Gronau and D. London, Phys. Lett. **B253**, 483 (1991).

[114] M. Gronau and D. Wyler, Phys. Lett. **B265**, 172 (1991).

[115] D. Atwood, I. Dunietz and A. Soni, Phys. Rev. Lett. **78**, 3257 (1997), [hep-ph/9612433]; D. Atwood, I. Dunietz and A. Soni, Phys. Rev. **D63**, 036005 (2001), [hep-ph/0008090].

[116] A. Bondar, talk at the Belle analysis workshop, Novosibirsk, September 2002; A. Poluektov *et al.* (Belle), Phys. Rev. **D70**, 072003 (2004), [hep-ex/0406067].

[117] A. Giri *et al.*, Phys. Rev. **D68**, 054018 (2003), [hep-ph/0303187].

[118] A. Poluektov *et al.* (Belle), Phys. Rev. **D81**, 112002 (2010), [arXiv:1003.3360].

[119] P. del Amo Sanchez *et al.* (BaBar), Phys. Rev. Lett. **105**, 121801 (2010), [arXiv:1005.1096].

[120] R. Aaij *et al.* (LHCb), JHEP **02**, 169 (2021), [arXiv:2010.08483].

[121] R. Aaij *et al.* (LHCb), JHEP **07**, 138 (2023), [arXiv:2209.03692]; R. Aaij *et al.* (LHCb), Eur. Phys. J. C **83**, 6, 547 (2023), [Erratum: Eur.Phys.J.C 83, 672 (2023)], [arXiv:2301.10328].

[122] Y. Grossman, A. Soffer and J. Zupan, Phys. Rev. **D72**, 031501 (2005), [hep-ph/0505270]; M. Rama, Phys. Rev. D **89**, 1, 014021 (2014), [arXiv:1307.4384].

[123] A. Amorim, M. G. Santos and J. P. Silva, Phys. Rev. **D59**, 056001 (1999), [hep-ph/9807364].

[124] R. Aleksan, I. Dunietz and B. Kayser, Z. Phys. **C54**, 653 (1992).

[125] R. Aaij *et al.* (LHCb), JHEP **03**, 059 (2018), [arXiv:1712.07428].

[126] R. Aaij *et al.* (LHCb), JHEP **03**, 137 (2021), [arXiv:2011.12041].

[127] B. Aubert *et al.* (BaBar), Phys. Rev. **D71**, 112003 (2005), [hep-ex/0504035]; B. Aubert *et al.* (BaBar), Phys. Rev. **D73**, 111101 (2006), [hep-ex/0602049]; F. J. Ronga *et al.* (Belle), Phys. Rev. **D73**, 092003 (2006), [hep-ex/0604013]; S. Bahinipati *et al.* (Belle), Phys. Rev. **D84**, 021101 (2011), [arXiv:1102.0888]; R. Aaij *et al.* (LHCb), JHEP **06**, 084 (2018), [arXiv:1805.03448].

[128] G. P. Dubois-Felsmann *et al.*, *Sensitivity of CKM fits to theoretical uncertainties and their representation* (2003), [hep-ph/0308262]; G. Eigen *et al.*, Phys. Rev. **D89**, 3, 033004 (2014), [arXiv:1301.5867].

[129] D. Boutigny *et al.* (BaBar), in "Workshop on Physics at an Asymmetric B Factory (BaBar Collaboration Meeting) Pasadena, California, September 22-24, 1997," (1998), URL `http://www-public.slac.stanford.edu/sciDoc/docMeta.aspx?slacPubNumber=SLAC-R-504`.

[130] S. Plaszczynski and M.-H. Schune hf8/019 (1999), [PoShf8,019(1999)], [hep-ph/9911280].

[131] M. Bona *et al.* (UTfit), JHEP **03**, 049 (2008), [arXiv:0707.0636].

[132] We thank the CKMfitter and UTfit groups for performing fits and preparing plots using input values from this *Review*.

[133] H. E. Haber, Nucl. Phys. Proc. Suppl. **62**, 469 (1998), [hep-ph/9709450]; Y. Nir, *CP violation: A New era* (2001), [hep-ph/0109090].

[134] J. M. Soares and L. Wolfenstein, Phys. Rev. **D47**, 1021 (1993); T. Goto *et al.*, Phys. Rev. **D53**, 6662 (1996), [hep-ph/9506311]; J. P. Silva and L. Wolfenstein, Phys. Rev. **D55**, 5331 (1997), [hep-ph/9610208].

[135] Y. Grossman, Z. Ligeti and Y. Nir, Prog. Theor. Phys. **122**, 125 (2009), [arXiv:0904.4262].

[136] G. Isidori, Y. Nir and G. Perez, Ann. Rev. Nucl. Part. Sci. **60**, 355 (2010), [arXiv:1002.0900]; G. Isidori, in "Proceedings, 2012 European School of High-Energy Physics (ESHEP 2012): La Pommeraye, Anjou, France, June 06-19, 2012," 69–105 (2014), [arXiv:1302.0661].

[137] Z. Ligeti, Int. J. Mod. Phys. **A20**, 5105 (2005), [hep-ph/0408267].

[138] J. Charles *et al.*, Phys. Rev. **D89**, 3, 033016 (2014), [arXiv:1309.2293].

[139] K. Agashe *et al.* (2005), [hep-ph/0509117].

[140] S. Laplace *et al.*, Phys. Rev. **D65**, 094040 (2002), [hep-ph/0202010].

[141] J. K. Ahn *et al.* (KOTO), Phys. Rev. Lett. **122**, 2, 021802 (2019), [arXiv:1810.09655]; J. K. Ahn *et al.* (KOTO), Phys. Rev. Lett. **126**, 12, 121801 (2021), [arXiv:2012.07571].

[142] A. J. Buras *et al.*, JHEP **11**, 033 (2015), [arXiv:1503.02693].