77. Determination of CKM angles from B hadrons

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77.1 Introduction

The Cabibbo–Kobayashi–Maskawa (CKM) description of quark mixing [1,2] leads to a number of triangle relations between pairs of CKM matrix elements. One of these,

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0 \;, \tag{77.1}$$

is of particular interest since (i) all its terms are of comparable magnitude, and (ii) its properties can be measured through studies of oscillations and decays of B mesons. As the area of this unitary triangle is a measure of the amount of CP violation in the Standard Model [3], it is of particular interest to determine the values of its angles and to test the consistency of the CKM paradigm with the experimental measurements. The angles are defined as

$$\alpha = \arg\left[-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right] \;, \quad \beta = \arg\left[-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right] \;, \quad \gamma = \arg\left[-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right] \;, \tag{77.2}$$

with an alternative notation $(\phi_2, \phi_1, \phi_3) \equiv (\alpha, \beta, \gamma)$ also widely used in the literature.

In this mini-review, the most precise methods to determine the CKM angles are described, with a particular focus on nontrivial aspects of the combination of results. More detailed discussions of these points can be found in Ref. [4]. A similar mini-review on the side of the unitarity triangle adjacent to the angle γ can be found in Ref. [5]. A detailed overview of the CKM quark-mixing matrix is given in Ref. [6] while CP violation in the quark sector is discussed in Ref. [7].

77.2 β

The relative weak (*i.e.* CP-violating) phase between the amplitude for any CKM-favoured B^0 meson decay to a CP eigenstate and that for the decay following B^0–$\overline{B}^0$ oscillation is twice the angle β. The decay-time-dependent CP asymmetry can be expressed as

$$\mathcal{A}_{f_{CP}}(t) \equiv \frac{d\Gamma/dt[\overline{B}^0_{\text{phys}}(t) \to f_{CP}] - d\Gamma/dt[B^0_{\text{phys}}(t) \to f_{CP}]}{d\Gamma/dt[\overline{B}^0_{\text{phys}}(t) \to f_{CP}] + d\Gamma/dt[B^0_{\text{phys}}(t) \to f_{CP}]} \;, \tag{77.3a}$$

$$= S_f \sin(\Delta m t) - C_f \cos(\Delta m t) \;, \tag{77.3b}$$

where the notation $B^0_{\text{phys}}(t)$ ($\overline{B}^0_{\text{phys}}(t)$) denotes a neutral B meson that decays at time t into the final state f_{CP}, and is known ("tagged") at time $t = 0$ to have flavour content corresponding to B^0 ($\overline{B}^0$). In Eq. (77.3b), Δm denotes the mass difference between the two physical eigenstates of the B^0–$\overline{B}^0$ system, while the corresponding decay-width difference is assumed to be negligible [8]; moreover CPT symmetry and the absence of CP violation in B^0–$\overline{B}^0$ mixing is assumed throughout this mini-review.

In the general case, one can write

$$S_f \equiv \frac{2\,\mathcal{I}m(\lambda_f)}{1+|\lambda_f|^2} \qquad \text{and} \qquad C_f \equiv \frac{1-|\lambda_f|^2}{1+|\lambda_f|^2} \;, \tag{77.4}$$

where the parameter $\lambda_f = \frac{q}{p}\frac{\overline{A}_f}{A_f}$ is defined in terms of p and q, which define the flavour content of the mass eigenstates of the B^0–$\overline{B}^0$ system [8], and the amplitudes $\overline{A}_f$ (A_f) for a $\overline{B}^0$ (B^0) decay to the final state f_{CP}. In the limit that the decay amplitude is dominated by a CKM-favoured transition, as is the case for $B^0 \to J/\psi K^0_{\rm S}$ decays, one obtains simple relations: $S_f = -\eta_{CP} \sin(2\beta)$ and $C_f = 0$, where η_{CP} is the *CP* eigenvalue of the final state [9,10]. This method has been pursued intensively by experiments. The current world averages, combining results for several charmonium-kaon final states but dominated by results on $B^0 \to J/\psi K^0_{\rm S}$ (*CP* odd) and $B^0 \to J/\psi K^0_{\rm L}$ (*CP* even), are [4]

$$-\eta_{CP} S_f = 0.709 \pm 0.011 \,, \qquad C_f = +0.004 \pm 0.010 \,. \tag{77.5}$$

Despite the large number of signal events in the data, the dominant uncertainties are still statistical. One important source of potential systematic correlation between results from different experiments is that due to “tag-side interference” [11],2 which is common to measurements exploiting production through the $e^+e^- \to \Upsilon(4S) \to B^0\overline{B}^0$ process, including the latest results from BaBar [12] and Belle [13]. It does not, however, affect the results from LHCb [14] that now have better statistical sensitivity. Another common source of systematic uncertainty is due to knowledge of the value of Δm, but since this quantity has been measured precisely [8] the effect remains small.

The interpretation of the value of $-\eta_{CP} S_f$ from Eq. (77.5) as $\sin(2\beta)$ assumes negligible contributions from subleading amplitudes with a different weak phase to that of the tree diagram (*i.e.* to that of the CKM matrix elements $V_{cb}V^*_{cs}$). This potential additional contribution is often referred to as “penguin pollution”. All existing data, including the value of C_f in Eq. (77.5), as well as several explicit calculations [15–18], are consistent with penguin pollution in B^0 meson decays to charmonium-kaon decays being negligible at the current level of precision. Therefore, the value of $-\eta_{CP} S_f$ is generally converted to $\sin(2\beta)$ without any correction or additional uncertainty being assigned due to this assumption. This gives [4]

$$\beta = \left(22.6\,^{+0.5}_{-0.4}\right)^\circ \,, \tag{77.6}$$

where only the solution consistent with the Standard Model is reported (methods to resolve the trigonometric ambiguity in the result are discussed below). It is also possible to use data-driven methods, typically based on flavour symmetries plus some additional assumptions, to constrain the effects of penguin pollution [19–21]. In this case it is necessary to consider each charmonium-kaon final state separately, since the penguin pollution to each may differ. The most common approach [19], which relies on experimental information on $B^0 \to J/\psi \pi^0$ decays, currently gives an additional uncertainty on $\sin(2\beta)$ from $B^0 \to J/\psi K^0_{\rm S}$ of around 0.01.

It is possible to avoid the issue of penguin pollution in the measurement of β by using B^0 meson decays to a charm- and light-meson final state, such as $D_{CP}\pi^0$ (where D_{CP} represents a D^0 meson decaying into a *CP* eigenstate), instead of the charmonium-kaon final states. These decays do have a CKM-suppressed contribution ($V_{ub}V^*_{cd}$ instead of $V_{cb}V^*_{ud}$), which can in principle bias the determination of $\sin(2\beta)$ from S_f, but this can be calculated and is known to be negligible at current precision. The requirement that the neutral D meson decays to a final state that is common to both D^0 and $\overline{D}^0$, such as the *CP*-even eigenstate K^+K^-, reduces the sample size that is available for analysis. Consequently, the world average [4], $\sin(2\beta) = 0.71 \pm 0.09$, with these channels is not as precise as that from the charmonium-kaon states.

Converting experimental results on $\sin(2\beta)$ into constraints on β leads to a trigonometric ambiguity in the range $[0^\circ, 180^\circ]$. This can be resolved with experimental measurements of $\cos(2\beta)$, which can be obtained from decay-time-dependent analyses of B^0 meson decays to multibody (non-*CP*-eigenstate) final states. Among the charmonium-kaon decays, study of $B^0 \to J/\psi K^*(892)^0$ with $K^*(892)^0 \to K^0_{\rm S}\pi^0$ is the most promising approach, but due to the limited sample size that has been analysed to date the precision is not sufficient to resolve the ambiguity conclusively. The charm- and light-meson channels such as $B^0 \to D\pi^0$ with $D \to K^0_S\pi^+\pi^-$ have been shown to provide good statistical power for this purpose, with a joint analysis of BaBar and Belle data giving $\cos(2\beta) = 0.91 \pm 0.25$ [22, 23], sufficient to rule out the alternative solution for β.

77.3 α

In the limit that only tree amplitudes contribute to B^0 meson decays to light mesons, such as $B^0 \to \pi^+\pi^-$, then the observables of the decay-time-dependent CP asymmetry of Eq. (77.3) would allow a straight-forward determination of 2α: $S_f = +\eta_{CP}\sin(2\alpha)$ and $C_f = 0$. In general, however, the determination of α is complicated by the presence of contributions from $b \to d(u\bar{u})$ neutral-current penguin transitions, which have a similar level of CKM-suppression as the $b \to u(\bar{u}d)$ charged-current tree amplitudes but have a different weak phase. Consequently, one obtains instead for $B^0 \to \pi^+\pi^-$

$$S_{\pi^+\pi^-} = \sqrt{1 - C^2_{\pi^+\pi^-}}\,\sin(2\alpha - 2\Delta\alpha), \tag{77.7}$$

where $\Delta\alpha$ is the *a priori* unknown penguin contribution.

This contribution from the penguin amplitude can be accounted for in an analysis relating the amplitudes for isospin partner decays, *e.g.* A^{+-} for $B^0 \to \pi^+\pi^-$, A^{+0} for $B^+ \to \pi^+\pi^0$, A^{00} for $B^0 \to \pi^0\pi^0$ decays and $\left(\overline{A}^{+-}, \overline{A}^{-0}, \overline{A}^{00}\right)$ for their charge conjugates. The isospin analysis relies on the fact that there is no penguin contribution to A^{+0} and $\overline{A}^{-0}$, because $\pi^\pm\pi^0$ is a pure isospin-2 state, and the $(\Delta I = \frac{1}{2})$ QCD-penguin amplitudes only contribute to the isospin-0 final state. One therefore obtains the following isospin triangle relations [25]

$$A^{+0} = \frac{1}{\sqrt{2}}A^{+-} + A^{00} \quad \text{and} \quad \overline{A}^{-0} = \frac{1}{\sqrt{2}}\overline{A}^{+-} + \overline{A}^{00}\,, \tag{77.8}$$

from which it is possible to determine $\Delta\alpha$, as shown in Fig. 77.1.

Since the determination of $\Delta\alpha$ and thus also α requires construction of amplitude-level relations, it is not appropriate to simply average results of α from different experiments. Instead, measurements of each of the observable quantities needed to determine α are input into a combination. For the $B \to \pi\pi$ system, the inputs are the branching fractions of $B^0 \to \pi^+\pi^-$, $B^+ \to \pi^+\pi^0$ and $B^0 \to \pi^0\pi^0$ decays, the lifetimes of the B^+ and B^0 mesons (which relate the branching fractions to amplitude-level quantities), and the $S_{\pi^+\pi^-}$, $C_{\pi^+\pi^-}$ and $C_{\pi^0\pi^0}$ observables. Potential sources of correlation must be taken into account, but these are predominantly systematic in origin and thus have a small effect on the combination, since the measurements are statistically limited. An exception is that the LHCb measurements of $(S_{\pi^+\pi^-}, C_{\pi^+\pi^-})$ [26, 27] have a significant statistical correlation due to the fact that the time variable of Eq. (77.3) is the difference between production and decay, and hence is in the range $[0, \infty]$. This correlation is largely absent for measurements from BaBar [28] and Belle [29], where the difference between the signal and tagging B meson decay times is measured, and hence $t \in [-\infty, \infty]$. The combination itself can be performed with different statistical approaches; the procedure described in detail in Ref. [30], based on a frequentist treatment, is used here. The knowledge of $C_{\pi^0\pi^0}$ [28, 31] is currently the limiting factor in the precision on α from the $B \to \pi\pi$ system, and is likely to remain so for some time due to the difficulty to reconstruct this final state.

In general, the isospin triangle construction gives a four-fold ambiguity on $2\Delta\alpha$ (each triangle can face either up or down), leading to an eight-fold ambiguity on α in the range $[0°, 180°]$. This is reduced if either or both of the triangles are flat, or if the two triangles have sides of identical length. The ambiguities can also be reduced if measurement of the $S_{\pi^0\pi^0}$ (or equivalent) observable is available, since this can be combined with the corresponding $\Delta\alpha$ parameter from the right-hand corner of the triangle in Fig. 77.1 to provide an additional constraint. None of these possibilities

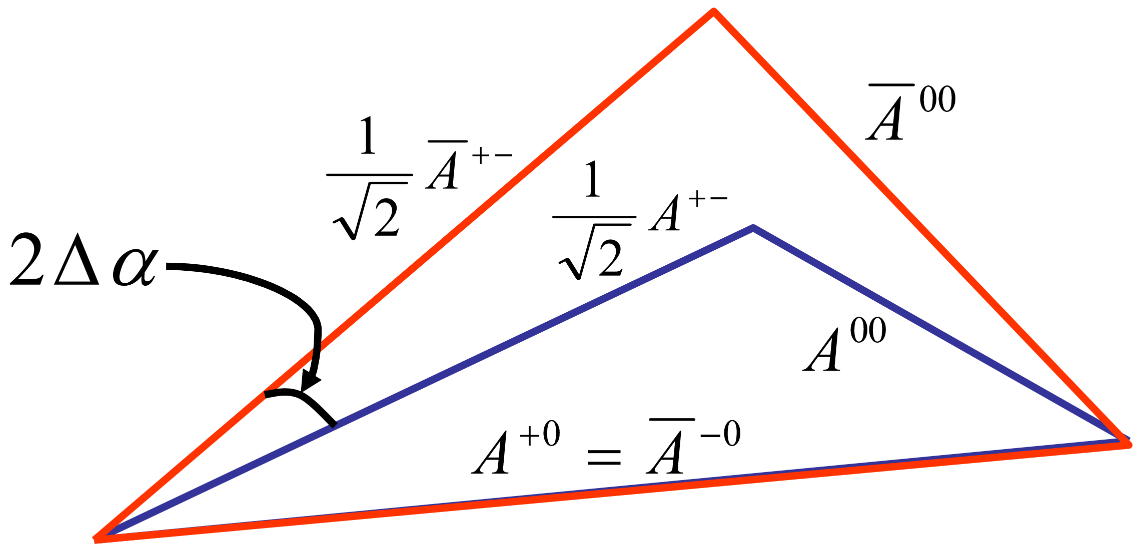

Figure 77.1: Isospin triangles for $B \to \pi\pi$ decays, reproduced from Ref. [24]. Here, the relative phase between A^{+0} and $\overline{A}^{-0}$ has been rotated away to simplify the picture. The total relative phase probed by $S_{\pi^+\pi^-}$ is $\arg\left(\frac{q}{p}\frac{\overline{A}^{+-}}{A^{+-}}\right) = 2\alpha - 2\Delta\alpha$, including contributions from B^0–$\overline{B}^0$ mixing, the tree-level amplitudes and the correction $\Delta\alpha$, and exploiting the unitarity requirement $\alpha+\beta+\gamma = 180°$.

are realised in the $B \to \pi\pi$ system; in particular a decay-time-dependent analysis of $B^0 \to \pi^0\pi^0$ is extremely challenging experimentally due to the absence of any charged particle originating from the B decay position. Nonetheless, solutions consistent with $\alpha = 0$ can be rejected on physical grounds [24].

The isospin analysis can also be performed with the $B \to \rho\rho$ system, which contains two vector particles in the final state and so does not have a fixed CP eigenvalue. In principle the analysis can be performed separately for each $\rho\rho$ polarization state, but in practise it is found that the longitudinal polarization fraction, f_L, is close to unity, and hence the final state is approximately CP-even. Compared to $B^0 \to \pi\pi$, the $\rho\rho$ modes benefit experimentally from a higher branching fraction and smaller penguin contributions, so that the isospin triangles are flatter, reducing the ambiguities. (The value of $\Delta\alpha$ in the $B \to \rho\rho$ system, obtained from the isospin analysis, has a single solution in $[0,\pi]$ at $(3 \pm 5)°$, while for $B \to \pi\pi$ there are two solutions at $13°$ and $27°$ with $\Delta\alpha \in [7,33]°$ at 68.3% confidence level (CL). The isospin analysis with either final state has an ambiguity under $\Delta\alpha \Leftrightarrow -\Delta\alpha$.) For the BaBar [32] and Belle [33] experiments, the high branching fraction and smaller penguin contribution compensate for the increased difficulty to reconstruct the $\rho\rho$ final state relative to $\pi\pi$. Moreover, in contrast to $S_{\pi^0\pi^0}$, measurement of $S_{\rho^0\rho^0}$ is possible due to the four charged pion final state, following $\rho^0 \to \pi^+\pi^-$ decay, as has been demonstrated by BaBar [34].

In the $B \to \rho\pi$ system there are more amplitudes to consider, so that the isospin relation corresponds to a pentagon rather than a triangle and Eq. (77.8) is modified to become

$$\sqrt{2}(A^{+0} + A^{0+}) = A^{+-} + A^{-+} + 2A^{00} \quad \text{and} \quad \sqrt{2}(\overline{A}^{-0} + \overline{A}^{0-}) = \overline{A}^{+-} + \overline{A}^{-+} + 2\overline{A}^{00} \,. \tag{77.9}$$

As in Eq. (77.8), the left-hand sides of these expressions correspond to a pure isospin-2 final state, and therefore the ratio of the right-hand sides gives a pure phase term that, accounting for the B^0–$\overline{B}^0$ mixing phase that also contributes to the measured quantities, is 2α. The relative amplitudes for B^0 and $\overline{B}^0$ decays to $\rho^+\pi^-$, $\rho^-\pi^+$ and $\rho^0\pi^0$ can all be determined from a decay-time-dependent

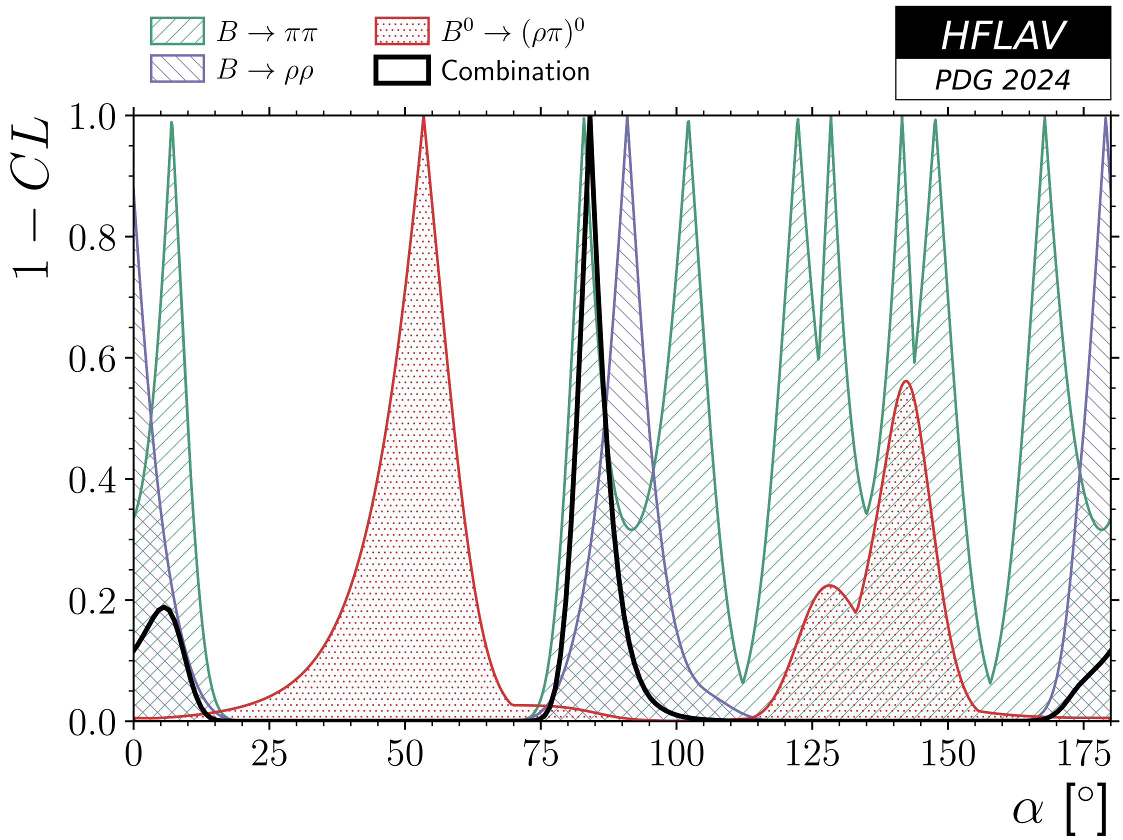

Figure 77.2: World average of α, as well as contributions from individual modes, in terms of 1−CL.

analysis of the $\pi^+\pi^-\pi^0$ Dalitz plot, so that study of this channel alone allows determination of α [35]. This analysis in principle leads to a single solution for α in $[0°, 180°]$, but the precision of current measurements [36–38] is limited.

The isospin analysis used to determine α is believed to be valid to high precision, and theoretical uncertainties in the procedure are usually neglected. Nonetheless, it should be noted that the analysis assumes the absence of electroweak penguin amplitudes, which can contribute to $\Delta I = \frac{3}{2}$ transitions with a different weak phase to that of the tree amplitudes [39, 40]. Moreover, isospin-breaking effects such as (π^0, η, η') mixing would impact on the relations of Eq. (77.8). A further complication in the $B \to \rho\rho$ system is the effect of the non-zero ρ meson width [41]. Estimates of the size of these effects on the determined value of α are typically at the 1° level or less [30]. By contrast, methods to determine α using SU(3) or other flavour symmetries are generally considered to have larger theoretical uncertainties and are not included here.

The world average obtained for the angle α from isospin analysis of $B \to \pi\pi$, $\rho\pi$ and $\rho\rho$ decays is [4]

$$\alpha = (84.1\,^{+4.5}_{-3.8})^{\circ}\,, \tag{77.10}$$

where the quoted uncertainty is at the 68.3% CL and does not include effects due to isospin-breaking. This world average, together with results split by decay mode, is shown in Fig. 77.2. The combination has a total of 57 experimental inputs from which 24 parameters are determined, and an overall χ^2 of 22.4, which corresponds to a p-value of 91%. Thus, there is excellent overall consistency between the inputs, despite the tension apparent in Fig. 77.2 between the results from

$B^0 \to (\rho\pi)^0$ and the others. The combination gives a single best-fit for α in $[0°, 180°]$, but an ambiguous solution exists at $\alpha \Leftrightarrow \alpha + 180°$. A secondary minimum close to zero is disfavoured [30].

77.4 γ

The angle γ is the weak phase between Cabibbo-favoured $b \to c$ and suppressed $b \to u$ quark transitions and can be determined by exploiting interference between them. Explicitly, the ratio of suppressed to favoured amplitudes is parameterized by

$$r_B e^{i(\delta_B \pm \gamma)} = \frac{A_{\rm sup}}{A_{\rm fav}} \,, \tag{77.11}$$

where r_B is the ratio of amplitude magnitudes, δ_B the strong phase difference and the $+$ or $-$ sign depends on whether the transition involves a $\bar{b}$ or b quark, respectively. Measurement of γ in this way has negligible theoretical uncertainty in the Standard Model [42], and therefore this approach provides a benchmark against which determinations from other methods, typically involving loop diagrams, can be compared.

Interference between these amplitudes is realised in $B^+ \to DK^+$ decays, where D represents an admixture of D^0 and $\overline{D}^0$ mesons. The simplest case is that of D decays to *CP*-eigenstates (GLW method [43, 44]), either *CP*-even such as K^+K^- (*CP*+) or *CP*-odd such as $K^0_{\rm S}\pi^0$ (*CP*−). The normalized decay rate and *CP* asymmetry are given by

$$R_{CP\pm} = \frac{\Gamma(B^- \to D_{CP\pm}K^-) + \Gamma(B^+ \to D_{CP\pm}K^+)}{\Gamma(B^- \to D^0K^-) + \Gamma(B^+ \to \overline{D}^0K^+)} = 1 + r_B^2 \pm 2r_B\cos(\delta_B)\cos(\gamma) \,, \tag{77.12a}$$

$$A_{CP\pm} = \frac{\Gamma(B^- \to D_{CP\pm}K^-) - \Gamma(B^+ \to D_{CP\pm}K^+)}{\Gamma(B^- \to D_{CP\pm}K^-) + \Gamma(B^+ \to D_{CP\pm}K^+)} = \frac{\pm 2r_B\sin(\delta_B)\sin(\gamma)}{1 + r_B^2 \pm 2r_B\cos(\delta_B)\cos(\gamma)} \,. \tag{77.12b}$$

These relations assume the absence of direct *CP* violation in the charm system; experimentally allowed deviations from this assumption are too small to cause a significant bias on γ [7, 45]. It is convenient to determine the $R_{CP\pm}$ quantities through a double ratio, normalizing to $B^+ \to D\pi^+$ decays involving the same final states, since this cancels potential sources of systematic uncertainty due to the branching fractions of the D decays that are used; small possible effects of *CP* violation in $B^+ \to D\pi^+$ decays are a source of systematic uncertainty in this procedure. The GLW method can be extended to include final states that are almost *CP*-eigenstates [46], as is the case in $D \to \pi^+\pi^-\pi^0$ and $D \to K^+K^-\pi^0$ decays, via inclusion of a factor encoding the fraction of *CP*-even (or *CP*-odd) content, $F_\pm$, which dilutes the sensitivity to γ by reducing the size of the interference terms (the terms linear with r_B) in Eq. (77.12).

For other D decays, the ratio of amplitudes for the D^0 and $\overline{D}^0$ decays to the final state of interest has to be accounted for in the formalism. The ADS method [47, 48] uses D decays to final states such as $K^\mp\pi^\pm$, which involve interference between Cabibbo-favoured (CF) and doubly-Cabibbo-suppressed (DCS) transitions. The observables in this case are

$$A_{\rm ADS} = \frac{\Gamma(B^- \to [K^+\pi^-]_D K^-) - \Gamma(B^+ \to [K^-\pi^+]_D K^+)}{\Gamma(B^- \to [K^+\pi^-]_D K^-) + \Gamma(B^+ \to [K^-\pi^+]_D K^+)} = \frac{2r_Br_D\sin(\delta_B + \delta_D)\sin(\gamma)}{r_B^2 + r_D^2 + 2r_Br_D\cos(\delta_B + \delta_D)\cos(\gamma)} \,, \tag{77.13a}$$

$$R_{\rm ADS} = \frac{\Gamma(B^- \to [K^+\pi^-]_D K^-) + \Gamma(B^+ \to [K^-\pi^+]_D K^+)}{\Gamma(B^- \to [K^-\pi^+]_D K^-) + \Gamma(B^+ \to [K^+\pi^-]_D K^+)} = r_B^2 + r_D^2 + 2r_Br_D\cos(\delta_B + \delta_D)\cos(\gamma) \,, \tag{77.13b}$$

where r_D and δ_D are the amplitude magnitude ratio and strong phase difference between the CF and DCS D decay. An alternative pair of observables, (R_-, R_+), is also sometimes used, where R_- (R_+)

is the ratio of decay rates between the suppressed and favoured transitions for B^- (B^+) decays. The R_- and R_+ observables are statistically independent, while $A_{\rm ADS}$ and $R_{\rm ADS}$ are not (in particular, the uncertainty on $A_{\rm ADS}$ depends on the central value of $R_{\rm ADS}$). However, the pair (R_-, R_+) has more correlated sources of systematic uncertainty compared to $(A_{\rm ADS}, R_{\rm ADS})$. The observables of Eq. (77.13) are therefore usually preferred once a significant signal is established. The ADS method can also be extended to include decays to multibody final states, such as $D \to K^\pm\pi^\mp\pi^0$ and $D \to K^\pm\pi^\mp\pi^+\pi^-$, by addition of a coherence factor [49] which appears in the interference terms of Eq. (77.13) and accounts for dilution of the sensitivity due to variation of the decay amplitude across the phase space of the final state. A similar method can be used for singly Cabibbo-suppressed D decays to non-*CP* eigenstates such as K^*K [50].

For D decays to multibody self-conjugate final states (BPGGSZ method [51,52]), such as $D \to K^0_{\rm S}\pi^+\pi^-$, one can write the partial decay rate as a function of the position in the phase space in terms of the "Cartesian parameters" $x_\pm + iy_\pm = r_B e^{i(\delta_B \pm \gamma)}$:

$$d\Gamma(B^\pm \to [K^0_{\rm S}\pi^+\pi^-]_D K^\pm) = A^2_{(\mp,\pm)} + r_B^2 A^2_{(\pm,\mp)} + 2A_{(\pm,\mp)}A_{(\mp,\pm)}[x_\pm c_{D(\pm,\mp)} + y_\pm s_{D(\pm,\mp)}], \tag{77.14}$$

where the notation $(+,-)$ is shorthand for the dependence on the Dalitz-plot position — the squared invariant masses of $K^0_{\rm S}\pi^+$ and $K^0_{\rm S}\pi^-$ combinations, respectively. The quantities $A_{(+,-)}$ and $A_{(-,+)}$ represent the magnitudes of the D^0 and $\overline{D}^0$ decay amplitudes at the position $(+,-)$ and are interchangable with their *CP* conjugate amplitudes because *CP* conservation is assumed in the D decay (*i.e.* $A_{(-,+)} = \overline{A}_{(+,-)}$). The quantities $c_{D(\pm,\mp)}$ and $s_{D(\pm,\mp)}$ are the cosine and sine of the strong phase difference, $\delta_{D(+,-)} = \arg(\overline{A}_{(+,-)}) - \arg(A_{(+,-)})$, between the $\overline{D}^0$ and D^0 amplitudes. These quantities can be determined from an amplitude model, although this leads to a hard-to-quantify systematic uncertainty associated to the composition of the model. An alternative, "model-independent", approach involves dividing the phase space into appropriate bins. In this case, the analysis benefits from external input on the values of c_D and s_D integrated over each bin. Measurements of these external parameters have been performed for the $D \to K^0_{\rm S}\pi^+\pi^-$ decay by the CLEO-c and BES-III collaborations [53–56]. The use of common input values for these parameters in model-independent determinations of γ with the BPGGSZ method by different experiments is a source of correlation between experiments that is currently negligible but will become more significant as the available B meson data samples increase in size.

The discussion above refers to $B^+ \to DK^+$ decays, but analogous measurements can be made also for additional channels such as $B^+ \to D^*K^+$ (with $D^* \to D\pi^0$, $D\gamma$) and $B^+ \to DK^{*+}$ (with $K^{*+} \to K^0_{\rm S}\pi^+$, $K^+\pi^0$). In the limit that these can be treated purely as two-body decays, the expressions for $B^+ \to DK^+$ are modified only by ensuring the r_B and δ_B parameters are specific to each B decay. Moreover, for $B^+ \to D^*K^+$ decays an effective shift of the strong phase by π between $D^* \to D\pi^0$ and $D\gamma$ decays [57] has to be taken into account. In case the finite width of the decaying resonance is non-negligible, as is the case for the $K^*(892)$ state, additional amplitudes can contribute leading to a dilution of the sensitivity, which can be accounted for in the formalism through the introduction of a relevant coherence factor. For the $B^0 \to DK^{*0}$ decay, full amplitude analysis of the $B^0 \to DK^+\pi^-$ Dalitz plot provides additional sensitivity compared to the quasi-two-body approach [58,59].

It is also possible to measure γ using decay-time-dependent analysis of the B^0_s meson [60]. The weak phase arising in the interference between direct decay of $B^0_s \to D^\mp_s K^\pm$ and decay via mixing is $(\gamma - 2\beta_s)$, where β_s is the angle associated with $B^0_s \to J/\psi\phi$ decays in a similar way to the relation between β and $B^0 \to J/\psi K^0_{\rm S}$ decays described in Sec. 77.2. Sufficient information can be obtained from the tagged, decay-time-dependent rates of $B^0_s \to D^\mp_s K^\pm$ decays that this weak phase can be determined, up to an ambiguity, together with the strong phase difference between, and the ratio of the magnitudes of, the suppressed and favoured amplitudes. Since β_s is known to good precision [8], measurements of the decay-time-dependent *CP*-asymmetry observables in $B_s^0 \to D_s^{\mp} K^{\pm}$ decays can be used to infer constraints on γ. Alternatively, if effects of penguin pollution in $B_s^0 \to J/\psi\phi$ decays [17, 18] are a concern, as they will become in the future, results from the $B_s^0 \to D_s^{\mp} K^{\pm}$ mode can be combined with an independent precise measurement of γ to provide a penguin-free determination of β_s.

The average for γ requires a non-trivial combination due the complicated relations between the observables and the physics parameters of interest, such as in Eqs. (77.12), (77.13) and (77.14). Moreover, hadronic parameters such as r_B and δ_B defined in Eq. (77.11) are common to all different D decay modes (but differ for each B decay mode). Thus, it is not correct to simply average results for γ obtained by different experiments or in different channels. Instead, measurements of rate asymmetries, rate ratios and the Cartesian parameters are taken as inputs to the combination, from which results are obtained not only for γ but also for the hadronic parameters. Independent measurements of auxiliary parameters such as r_D and δ_D are also treated as inputs to the combination. In some cases the B decay data can help to reduce uncertainties on these auxiliary parameters and therefore a simultaneous fit of charm and beauty data can provide stronger constraints [61]; this approach however is not currently used for the world average.

The precision to which γ can be measured with a particular B decay is approximately inversely proportional to the value of r_B. Thus, results from channels with smaller yields but larger values of r_B, such as $B^0 \to DK^{*0}$ and $B_s^0 \to D_s^{\mp} K^{\pm}$ ($r_B \approx 0.3$–0.4), can have a significant impact on the world average and are included in the combination. By contrast the $B^+ \to D\pi^+$ mode, for which large samples are available but $r_B \approx 0.005$, has little impact and is also more sensitive to potential systematic biases; hence it is not included. The sensitivity of the world average at present is dominated by results from $B^+ \to DK^+$, where $r_B \approx 0.1$, in particular results with the GLW [62], ADS [62] and BPGGSZ [63] methods.

The world average obtained for the angle γ, obtained by combining results from $B^+ \to DK^+$, D^*K^+, DK^{*+}, $DK^+\pi^+\pi^-$, $B^0 \to DK^+\pi^-$, $B_s^0 \to D_s^{\mp} K^{\pm}$ and $B_s^0 \to D_s^{\mp} K^{\pm}\pi^+\pi^-$ decays, is [4]

$$\gamma = (65.7 \pm 3.0)^\circ \ , \tag{77.15}$$

where the quoted uncertainty is at the 68.3% CL.

Effects related to charm and kaon mixing and *CP* violation are generally negligible at the current level of precision, in particular for modes with $r_B \gtrsim 0.1$. An exception is that a dependence of the selection efficiency on the charm decay time can induce a dependence of the observables on charm mixing parameters [64]. Such effects can be important at hadron collider experiments such as LHCb, but can be and are corrected for. Interactions of neutral kaons with detector material can also cause a bias in determination of γ from modes with low values of r_B [65], such as the BPGGSZ method applied to $B^+ \to D\pi^+$, but are negligible in modes with larger r_B values. A further subtlety is that the identification of the weak phase between suppressed and favoured amplitudes in $B \to DK$ decays with γ, as defined in Eq. (77.2), assumes that the 2×2 submatrix of the CKM matrix is real, *i.e.* that $\arg\left[V_{ud}V_{us}^* / \left(V_{cd}V_{cs}^*\right)\right] = 0$. This is true to an excellent approximation in the Standard Model, and is known experimentally from independent studies of the charm system [45] to contribute negligible bias to current measurements. Nonetheless, in future it will be possible to test directly this assumption by comparing the value of γ obtained from the $B \to DK$ and $B \to D\pi$ systems.

Effects from correlated uncertainties between amplitude models and strong phase differences in charm decays are negligible and are not explicitly accounted for in the combination, nor are effects

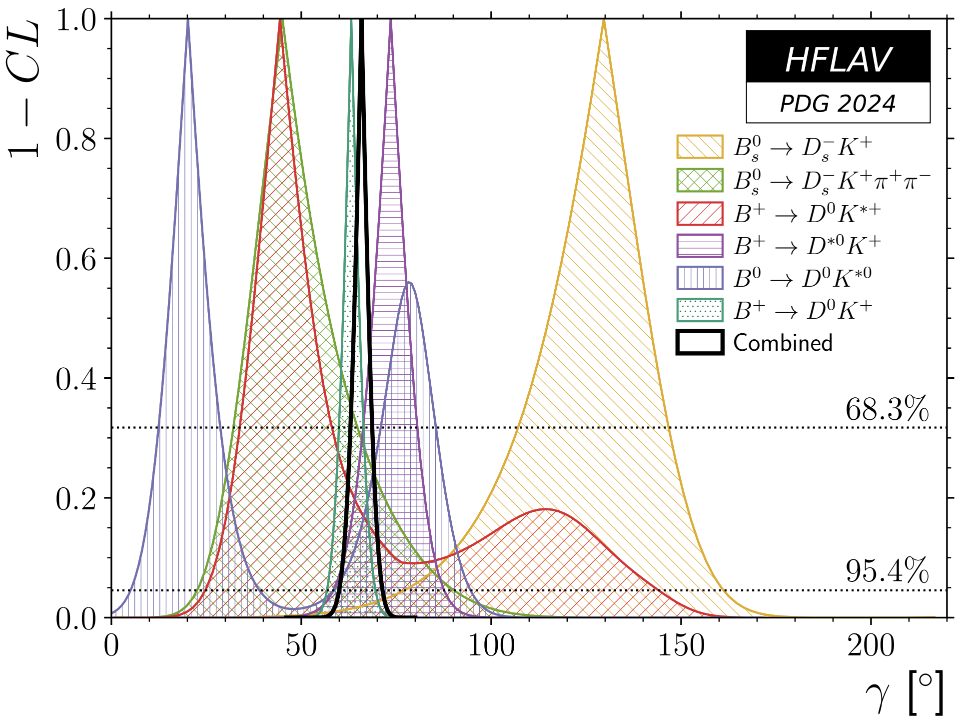

Figure 77.3: World average of $\gamma \equiv \phi_3$, as well as contributions from individual modes, in terms of 1−CL.

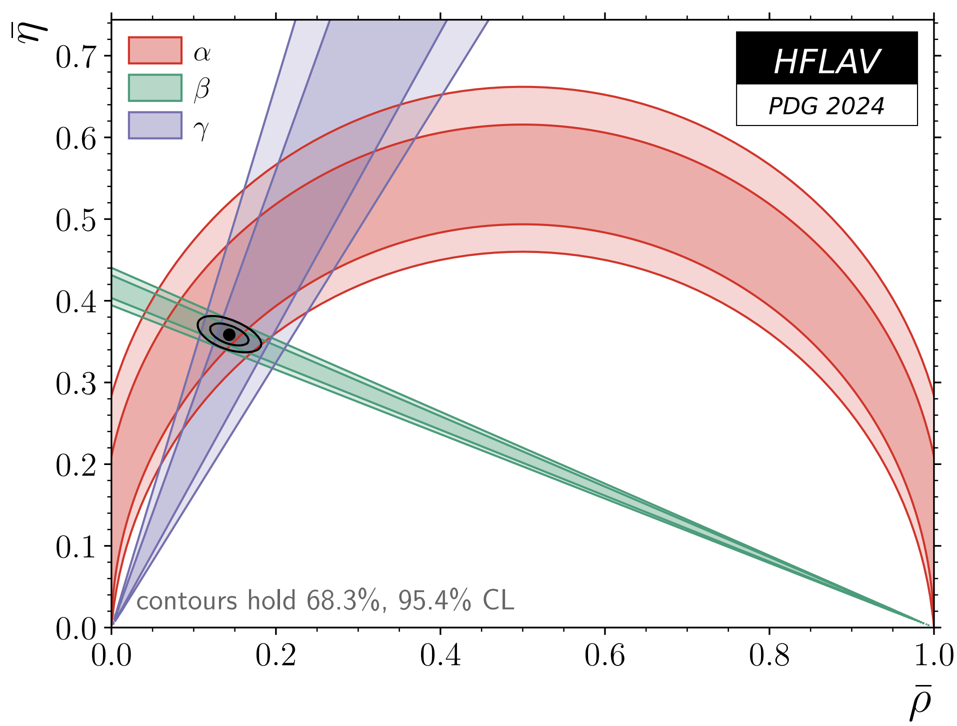

Figure 77.4: Constraints from the measurements of the angles of the CKM unitarity triangle in the $(\overline{\rho}, \overline{\eta})$ plane.

related to charm and kaon mixing and *CP* violation. This world average, together with results split by decay mode, is shown in Fig. 77.3. The combination has a total of 173 experimental inputs from which 36 parameters are determined, an an overall χ^2 of 159.5, which corresponds to a p-value of 9% indicating reasonable agreement between the inputs. The combination gives a single solution for γ in $[0°, 180°]$, but an ambiguous solution exists at $\gamma \Leftrightarrow \gamma + 180°$.

77.5 Summary

Experimental progress has resulted in all three angles of the CKM unitarity triangle being measured with good accuracy, with β known to subdegree precision and both α and γ known to better than 5°. The constraints from these three measurements in the $(\overline{\rho}, \overline{\eta})$ plane are shown in Fig. 77.4; further discussion and comparison with constraints from independent measurements can be found in Ref. [6]. The determinations of all three angles remain statistically limited, but it will be a challenge for experiments to ensure that this remains the case as the precision improves. Consequently, the correct treatment of sources of correlation between the measurements that go into the world average combinations is becoming increasingly important.

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