

### 44. Clebsch-Gordan Coefficients, Spherical Harmonics, and $d$ Functions

Note: A square-root sign is to be understood over every coefficient, e.g., for  $-8/15$  read  $-\sqrt{8/15}$ .

Notation:

$J$	$J$	...
$M$	$M$	...

$$1/2 \times 1/2$$

1	0
+1/2	1/2
-1/2	1/2
0	0
0	0
-1/2	-1/2
1	0

$$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$$

$$Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}$$

$$2 \times 1/2$$

5/2	3/2
+5/2	1
+2+1/2	1/5 4/5
+1+1/2	4/5-1/5
0+1/2	5/2 3/2
	3/5-2/5
	-1/2-1/2

$m_1$	$m_2$	Coefficients
$m_1$	$m_2$	
$\vdots$	$\vdots$	
$\vdots$	$\vdots$	

$$1 \times 1/2$$

3/2	1/2
+3/2	1
+1-1/2	1/3 2/3
0+1/2	2/3-1/3
	0-1/2
	2/3 1/3
	-1+1/2
	1/3-2/3
	3/2
	-3/2

$$3/2 \times 1/2$$

2	1
+2	1
+3/2+1/2	1
+1+1/2	1
	2 1
	3/2-1/2
	1/4 3/4
	3/4-1/4
	0 0

$$2 \times 1$$

3	2
+3	2
+2+1	1
	2 2
	2 1
	1/3 2/3
	2/3-1/3
	3 2 1
	+1 +1 +1

$$3/2 \times 1$$

5/2	3/2
+5/2	1
+3/2+1	1
	3/2 3/2
	2/5 3/5
	3/5-2/5
	5/2 3/2 1/2
	+1/2+1/2

$$1 \times 1$$

2	1
+2	1
+1+1	1
	2 1
	1/2 1/2
	1/2-1/2
	2 1 0
	0 0 0

3	2	1
3	2	1
0 0 0	1	0
0 0 0	1	0
0 0 0	1	0
0 0 0	1	0
0 0 0	1	0
0 0 0	1	0
0 0 0	1	0

5/2	3/2	1/2
+5/2	1	1
+3/2-1	1/10 2/5 1/2	
+1/2 0	3/5 1/15 -1/3	
-1/2+1	3/10-8/15 1/6	
	5/2 3/2 1/2	
	-1/2 -1/2 -1/2	

2	1
2	1
1/2 1/2	2 1
1/2-1/2	-1 -1
-1/2-1/2	3/4 1/4
-3/2+1/2	1/4-3/4
	-1/2-1/2
	3/4 1/4
	-2

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d_{m,0}^\ell = \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M \rangle = (-1)^{J-j_1-j_2} \langle j_2 j_1 m_2 m_1 | j_2 j_1 J M \rangle$$

$$d_{m',m}^j = (-1)^{m-m'} d_{m,m'}^j = d_{-m,-m'}^j$$

$$3/2 \times 3/2$$

3	2
+3	2
+3/2+3/2	1
	3 2
	1/2 1/2
	1/2-1/2
	+1 +1 +1

$$d_{0,0}^1 = \cos \theta$$

$$d_{1/2,1/2}^{1/2} = \cos \frac{\theta}{2}$$

$$d_{1,1}^1 = \frac{1 + \cos \theta}{2}$$

$$d_{1/2,-1/2}^{1/2} = -\sin \frac{\theta}{2}$$

$$d_{1,0}^1 = -\frac{\sin \theta}{\sqrt{2}}$$

$$d_{1,-1}^1 = \frac{1 - \cos \theta}{2}$$

$$2 \times 3/2$$

7/2	5/2
+7/2	1
+2+3/2	1
	7/2 5/2
	3/7 4/7
	4/7-3/7
	7/2 5/2 3/2
	+3/2+3/2

$$2 \times 2$$

4	3
+4	3
+2+2	1
	4 3
	1/2 1/2
	1/2-1/2
	+2 +2 +2

7/2	5/2	3/2
+7/2	1	1
+2+1/2	1/7 16/35 2/5	
+1+3/2	4/7 1/35 -2/5	
0+3/2	2/7-18/35 1/5	
	7/2 5/2 3/2 1/2	
	+1/2+1/2+1/2+1/2	

3	2	1
3	2	1
+3/2-1/2	1/5 1/2 3/10	
+1/2+1/2	3/5 0 -2/5	
-1/2+3/2	1/5-1/2 3/10	
	3 2 1	
	0 0 0	
	0 0 0	

3/2	3/2	1/2
+3/2	1	1
+1/2-3/2	1/20 1/4 9/20 1/4	
+1/2-1/2	9/20 1/4-1/20-1/4	
-1/2+1/2	9/20-1/4-1/20 1/4	
-3/2+3/2	1/20-1/4 9/20-1/4	
	3 2 1	
	-1 -1 -1	

3	2	1
3	2	1
+1/2-3/2	1/5 1/2 3/10	
-1/2-1/2	3/5 0 -2/5	
-3/2+1/2	1/5-1/2 3/10	
	3 2	
	-2 -2	

3	2	1
3	2	1
-1/2-3/2	1/2 1/2 3	
-3/2-1/2	1/2-1/2 -3	
-3/2-3/2	1	

$$d_{3/2,3/2}^{3/2} = \frac{1 + \cos \theta}{2} \cos \frac{\theta}{2}$$

$$d_{3/2,1/2}^{3/2} = -\sqrt{3} \frac{1 + \cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{3/2,-1/2}^{3/2} = \sqrt{3} \frac{1 - \cos \theta}{2} \cos \frac{\theta}{2}$$

$$d_{3/2,-3/2}^{3/2} = -\frac{1 - \cos \theta}{2} \sin \frac{\theta}{2}$$

$$d_{1/2,1/2}^{3/2} = \frac{3 \cos \theta - 1}{2} \cos \frac{\theta}{2}$$

$$d_{1/2,-1/2}^{3/2} = -\frac{3 \cos \theta + 1}{2} \sin \frac{\theta}{2}$$

$$d_{2,2}^2 = \left( \frac{1 + \cos \theta}{2} \right)^2$$

$$d_{2,1}^2 = -\frac{1 + \cos \theta}{2} \sin \theta$$

$$d_{2,0}^2 = \frac{\sqrt{6}}{4} \sin^2 \theta$$

$$d_{2,-1}^2 = -\frac{1 - \cos \theta}{2} \sin \theta$$

$$d_{2,-2}^2 = \left( \frac{1 - \cos \theta}{2} \right)^2$$

$$d_{1,1}^2 = \frac{1 + \cos \theta}{2} (2 \cos \theta - 1)$$

$$d_{1,0}^2 = -\sqrt{\frac{3}{2}} \sin \theta \cos \theta$$

$$d_{1,-1}^2 = \frac{1 - \cos \theta}{2} (2 \cos \theta + 1)$$

$$d_{0,0}^2 = \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

Figure 44.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).