

44. Clebsch-Gordan Coefficients, Spherical Harmonics, and d Functions

Note: A square-root sign is to be understood over every coefficient, e.g., for $-8/15$ read $-\sqrt{8/15}$.

Notation:

J	J	...
M	M	...
m_1	m_2	
m_1	m_2	Coefficients
\vdots	\vdots	
\vdots	\vdots	

$1/2 \times 1/2$

1		
+1/2	+1/2	1
+1/2	-1/2	1/2
-1/2	+1/2	1/2
-1/2	-1/2	1

$1 \times 1/2$

3/2			
+3/2	3/2	1/2	
+1	+1/2	1	
+1	-1/2	1/3	2/3
0	+1/2	2/3	-1/3
0	-1/2	2/3	1/3
-1	+1/2	1/3	-2/3
-1	-1/2	3/2	

2×1

3			
+3	3	2	
+2	0	1/3	2/3
+1	+1	2/3	-1/3
0	-1/2	2/3	1/3
-1	+1/2	1/3	-2/3
-1	-1/2	3/2	

$3/2 \times 1$

5/2			
+5/2	5/2	3/2	
+3/2	+1	1	
+3/2	0	2/5	3/5
+1/2	+1	3/5	-2/5
0	-1/2	3/5	2/5
-1/2	+1/2	1/5	-4/5
-1/2	-1/2	5/2	

$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$

$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$

$Y_2^0 = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$

$Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$

$Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}$

$3/2 \times 1/2$

2			
+3/2	+1/2	1	
+2	0	1/4	3/4
+1	+1	3/4	-1/4
0	-1/2	1/2	1/2
-1	+1/2	1/2	-1/2
-1	-1/2	5/2	

$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$

$d_{m,0}^\ell = \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m e^{-im\phi}$

$(j_1 j_2 m_1 m_2 | j_1 j_2 JM)$
 $= (-1)^{J-j_1-j_2} (j_2 j_1 m_2 m_1 | j_2 j_1 JM)$

$d_{m',m}^j = (-1)^{m-m'} d_{m,m'}^j = d_{-m,-m'}^j$

$2 \times 3/2$

7/2			
+7/2	7/2	5/2	
+2	+3/2	1	
+2	+1/2	3/7	4/7
+1	+3/2	4/7	-3/7
0	-1/2	7/2	5/2
0	+3/2	3/2	3/2
-1	+1/2	1/7	16/35
-1	+3/2	2/7	-18/35
-2	-1/2	5/2	3/2

$d_{1,0}^1 = \cos \theta$

$d_{1/2,1/2}^{1/2} = \cos \frac{\theta}{2}$

$d_{1/2,-1/2}^{1/2} = -\sin \frac{\theta}{2}$

$d_{1,1}^1 = \frac{1 + \cos \theta}{2}$

$d_{1,0}^1 = -\frac{\sin \theta}{\sqrt{2}}$

$d_{1,-1}^1 = \frac{1 - \cos \theta}{2}$

$3/2 \times 3/2$

3			
+3	3	2	
+3/2	+3/2	1	
+3/2	+1/2	1/2	1/2
+1	+1	1	1
0	-1/2	1/5	1/2
0	+3/2	3/5	0
-1	+1/2	1/5	-1/2
-1	-1/2	3/10	3/10
-2	-1/2	3/2	1/2

$d_{3/2,3/2}^{3/2} = \frac{1 + \cos \theta}{2} \cos \frac{\theta}{2}$

$d_{3/2,1/2}^{3/2} = -\sqrt{3} \frac{1 + \cos \theta}{2} \sin \frac{\theta}{2}$

$d_{3/2,-1/2}^{3/2} = \sqrt{3} \frac{1 - \cos \theta}{2} \cos \frac{\theta}{2}$

$d_{3/2,-3/2}^{3/2} = -\frac{1 - \cos \theta}{2} \sin \frac{\theta}{2}$

$d_{1/2,1/2}^{3/2} = \frac{3 \cos \theta - 1}{2} \cos \frac{\theta}{2}$

$d_{1/2,-1/2}^{3/2} = -\frac{3 \cos \theta + 1}{2} \sin \frac{\theta}{2}$

$d_{2,2}^2 = \left(\frac{1 + \cos \theta}{2} \right)^2$

$d_{2,1}^2 = -\frac{1 + \cos \theta}{2} \sin \theta$

$d_{2,0}^2 = \frac{\sqrt{6}}{4} \sin^2 \theta$

$d_{2,-1}^2 = -\frac{1 - \cos \theta}{2} \sin \theta$

$d_{2,-2}^2 = \left(\frac{1 - \cos \theta}{2} \right)^2$

$d_{1,1}^2 = \frac{1 + \cos \theta}{2} (2 \cos \theta - 1)$

$d_{1,0}^2 = -\sqrt{\frac{3}{2}} \sin \theta \cos \theta$

$d_{1,-1}^2 = \frac{1 - \cos \theta}{2} (2 \cos \theta + 1)$

$d_{0,0}^2 = \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$

Figure 44.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).