V_{ud}, V_{us}, THE CABIBBO ANGLE, AND CKM UNITARITY

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The Cabibbo-Kobayashi-Maskawa (CKM) [1,2] three-generation quark mixing matrix written in terms of the Wolfenstein parameters (λ, A, ρ, η) [3] nicely illustrates the orthonormality constraint of unitarity and central role played by λ.

$$V_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)\ . \quad (1)$$

That cornerstone is a carryover from the two-generation Cabibbo angle, $\lambda = \sin(\theta_{\rm Cabibbo}) = V_{us}$. Its value is a critical ingredient in determinations of the other parameters and in tests of CKM unitarity.

Until about 11 years ago, the precise value of λ was somewhat controversial, with kaon decays suggesting [4] $\lambda \simeq 0.220$, while indirect determinations via nuclear β-decays implied a somewhat larger $\lambda \simeq 0.225 - 0.230$. This difference resulted in a $2 - 2.5$ sigma deviation from the unitarity requirement

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1, \quad (2)$$

a potential signal [5] for new physics effects. Below, we discuss the current status of V_{ud}, V_{us}, and their associated unitarity test in Eq. (2). (Since $|V_{ub}|^2 \simeq 1 \times 10^{-5}$ is negligibly small, it is ignored in this discussion.) Eq. (2) is currently the most stringent test of unitarity in the CKM matrix.

V_{ud}

The value of V_{ud} has been obtained from superallowed nuclear, neutron, and pion decays. Currently, the most precise determination of V_{ud} comes from a set of superallowed nuclear beta-decays [5] ($0^+ \to 0^+$ transitions). Measuring their half-lives, t, and Q values that give the decay rate factor, f, leads to a precise determination of V_{ud} via the master formula [6–10]

$$|V_{ud}|^2 = \frac{2984.48(5)\,\mathrm{sec}}{ft(1+\Delta)}, \tag{3}$$

where Δ denotes the entire effect of electroweak radiative corrections (RC), nuclear structure, and isospin violating nuclear effects. Δ is nucleus-dependent, ranging from about +3.0% to +3.6% for the best measured superallowed decays.

The most recent analysis of 14 precisely measured superallowed transitions by Hardy and Towner [11] gives a weighted average of

$$V_{ud} = 0.97417(5)_{\mathrm{exp.}}(9)_{\mathrm{nucl.dep.}}(18)_{\mathrm{RC}} \text{ (superallowed)} , \tag{4}$$

which, assuming unitarity, corresponds to $\lambda = 0.2258(9)$. This recent determination of V_{ud} has shifted downward compared to the 2014 value of 0.97425(22) primarily from improvements in the nuclear isospin breaking corrections [11]. It is now closer to the central value quoted in 2007.

Combined measurements of the neutron lifetime, τ_n, and the ratio of axial-vector/vector couplings, $g_A \equiv G_A/G_V$, via neutron decay asymmetries can also be used to determine V_{ud}:

$$|V_{ud}|^2 = \frac{4908.7(1.9)\ \mathrm{sec}}{\tau_n(1+3g_A^2)}, \tag{5}$$

where the error stems from uncertainties in the electroweak radiative corrections [7] due to hadronic loop effects. Those effects were updated and their error was reduced by about a factor of 2 [8], leading to a ±0.0002 theoretical uncertainty in V_{ud} (common to all V_{ud} extractions). Using the world averages from this *Review*

$$\begin{aligned} \tau_n^{\mathrm{ave}} &= 880.3(1.1)\ \mathrm{sec}\ \ (\times 1.9\ \text{PDG scale factor}) \\ g_A^{\mathrm{ave}} &= 1.2723(23)\ \ (\times 2.2\ \text{PDG scale factor}) \end{aligned} \tag{6}$$

leads to

$$V_{ud} = 0.9758(6)_{\tau_n}(15)_{g_A}(2)_{\mathrm{RC}}, \tag{7}$$

with the error dominated by g_A uncertainties. We note that the larger g_A now adopted in Eq. (6) leads to a value of V_{ud} that is still somewhat high, but in accord with the superallowed nuclear beta decay result in Eq. (4). Future neutron studies [12] are expected to resolve any current inconsistencies and significantly reduce the uncertainties in g_A and τ_n, potentially making them a competetive way to determine V_{ud} without nuclear physics uncertainties.

The PIBETA experiment at PSI measured the very small ($\mathcal{O}(10^{-8})$) branching ratio for $\pi^+ \to \pi^o e^+ \nu_e$ with about $\pm 1/2\%$ precision. Their result gives [13]

$$V_{ud} = 0.9749(26) \left[\frac{BR(\pi^+ \to e^+ \nu_e(\gamma))}{1.2352 \times 10^{-4}} \right]^{\frac{1}{2}} \tag{8}$$

which is normalized using the very precisely determined theoretical prediction for $BR(\pi^+ \to e^+\nu_e(\gamma)) = 1.2352(5) \times 10^{-4}$ [6], rather than the experimental branching ratio from this *Review* of $1.230(4) \times 10^{-4}$ which would lower the value to $V_{ud} = 0.9728(30)$. Theoretical uncertainties in the pion β-decay determination are very small; however, much higher statistics would be required to make this approach competitive with others.

V_{us}

$|V_{us}|$ may be determined from kaon decays, hyperon decays, and tau decays. Previous determinations have most often used $K\ell 3$ decays:

$$\Gamma_{K\ell 3} = \frac{G_F^2 M_K^5}{192\pi^3} S_{EW}(1 + \delta_K^\ell + \delta_{SU2}) C^2 \left|V_{us}\right|^2 f_+^2(0) I_K^\ell . \tag{9}$$

Here, ℓ refers to either e or μ, G_F is the Fermi constant, M_K is the kaon mass, S_{EW} is the short-distance radiative correction, δ_K^ℓ is the mode-dependent long-distance radiative correction, $f_+(0)$ is the calculated form factor at zero momentum transfer for the $\ell\nu$ system, and I_K^ℓ is the phase-space integral, which depends on measured semileptonic form factors. For charged kaon decays, δ_{SU2} is the deviation from one of the ratio of $f_+(0)$ for the charged to neutral kaon decay; it is zero for the neutral kaon. C^2 is 1 (1/2) for neutral (charged) kaon decays. Most early determinations of $|V_{us}|$ were based soley on

$K \to \pi e \nu$ decays; $K \to \pi \mu \nu$ decays were not used because of large uncertainties in I_K^μ. The experimental measurements are the semileptonic decay widths (based on the semileptonic branching fractions and lifetime) and form factors (allowing calculation of the phase space integrals). Theory is needed for S_{EW}, δ_K^ℓ, δ_{SU2}, and $f_+(0)$.

Many measurements during the last decade have resulted in a significant shift in V_{us}. Most importantly, recent measurements of the $K \to \pi e \nu$ branching fractions are significantly different than earlier PDG averages, probably as a result of inadequate treatment of radiation in older experiments. This effect was first observed by BNL E865 [14] in the charged kaon system and then by KTeV [15,16] in the neutral kaon system; subsequent measurements were made by KLOE [17–20], NA48 [21–23], and ISTRA+ [24]. Current averages (*e.g.*, by the PDG [25] or Flavianet [26]) of the semileptonic branching fractions are based only on recent, high-statistics experiments where the treatment of radiation is clear. In addition to measurements of branching fractions, new measurements of lifetimes [27] and form factors [28–32], have resulted in improved precision for all of the experimental inputs to V_{us}. Precise measurements of form factors for $K_{\mu 3}$ decay make it possible to use both semileptonic decay modes to extract V_{us}.

Following the analysis of Moulson [33] and the Flavianet group [26], one finds, after including the isospin violating up-down mass difference effect, the values of $|V_{us}|f_+(0)$ in Table 1. The average of these measurements gives

$$f_+(0)|V_{us}| = 0.2165(4). \tag{10}$$

Figure 1 shows a comparison of these results with the PDG evaluation from 2002 [34], as well as $f_+(0)(1-|V_{ud}|^2-|V_{ub}|^2)^{1/2}$, the expectation for $f_+(0)|V_{us}|$ assuming unitarity, based on $|V_{ud}| = 0.97417 \pm 0.00021$, and $|V_{ub}| = (4.1 \pm 0.4) \times 10^{-3}$ [35].

Lattice calculations of $f_+(0)$ have been carried out for 2, 2+1, and 2+1+1 quark flavors and range from about 0.96 to 0.97. Here, we use $f_+(0) = 0.9677(37)$, the 2015 preliminary (2+1)-flavor FLAG average reported by Rosner, Stone, and Van de Water in footnote 10 of their PDG review of pseudoscalar decay constants [35], in Eq. (10), and find

$$|V_{us}| = \lambda = 0.2237(4)_{\rm exp+RC}(9)_{\rm lattice}\ (K_{\ell 3}\ \text{Decays})\ . \qquad (11)$$

Table 1: $|V_{us}|f_+(0)$ from $K_{\ell 3}$.

Decay Mode	$\lvert V_{us}\rvert f_+(0)$
$K^{\pm}e3$	0.2172 ± 0.0008
$K^{\pm}\mu 3$	0.2170 ± 0.0011
K_Le3	0.2163 ± 0.0006
$K_L\mu 3$	0.2166 ± 0.0006
K_Se3	0.2155 ± 0.0013
Average	0.2165 ± 0.0004

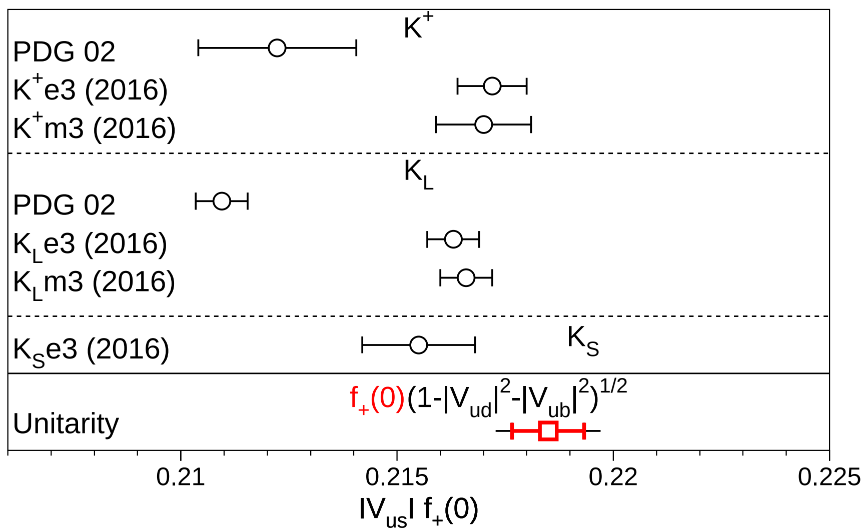

Figure 1: Comparison of determinations of $|V_{us}|f_+(0)$ from this review (labeled 2016), from the PDG 2002, and with the prediction from unitarity using $|V_{ud}|$ and the lattice calculation of $f_+(0) = 0.9677(37)$ [35]. For $f_+(0)(1-|V_{ud}|^2-|V_{ub}|^2)^{1/2}$, the inner error bars are from the quoted uncertainty in $f_+(0)$; the total uncertainties include the $|V_{ud}|$ and $|V_{ub}|$ errors.

A value of V_{us} can also be obtained from a comparison of the radiative inclusive decay rates for $K \to \mu\nu(\gamma)$ and $\pi \to \mu\nu(\gamma)$ combined with a lattice gauge theory calculation of f_{K^+}/f_{π^+} via [42]

$$\frac{|V_{us}|f_{K^+}}{|V_{ud}|f_{\pi^+}} = 0.23871(20) \left[\frac{\Gamma(K \to \mu\nu(\gamma))}{\Gamma(\pi \to \mu\nu(\gamma))}\right]^{\frac{1}{2}} \tag{12}$$

with the small error coming from electroweak radiative corrections and isospin breaking effects. Employing

$$\frac{\Gamma(K \to \mu\nu(\gamma))}{\Gamma(\pi \to \mu\nu(\gamma))} = 1.3367(29), \tag{13}$$

which includes the recent update $\Gamma(K \to \mu\nu(\gamma)) = 5.134(11) \times 10^7 s^{-1}$ [33,43] and [35]

$$f_{K^+}/f_{\pi^+} = 1.1928(26) \tag{14}$$

along with the value of V_{ud} in Eq. (4) leads to

$$|V_{us}| = 0.22540(53)_{\text{exp}}(19)_{\text{RC}}(49)_{\text{lattice}} \ (K_{\mu 2} \text{ Decays}) \ . \tag{15}$$

Together, a weighted average of the $K\ell3$ (Eq. (11)) and $K\mu2$ (Eq. (15)) results gives

$$|V_{us}| = 0.2248(6). \tag{16}$$

It should be mentioned that hyperon decay fits suggest [45]

$$|V_{us}| = 0.2250(27) \text{ (Hyperon Decays)} \tag{17}$$

modulo SU(3) breaking effects that could shift that value up or down. We note that a representative effort [46] that incorporates SU(3) breaking found $V_{us} = 0.226(5)$. Strangeness changing tau decays, averaging both inclusive and exclusive measurements, currently give [47]

$$|V_{us}| = 0.2202(15) \text{ (Tau Decays)} \ , \tag{18}$$

which differs by about 3 sigma from the kaon determination discussed above, and would, if combined with V_{ud} from super-allowed beta decays, lead to a 2.6 sigma deviation from unitarity. This discrepancy results mainly from the inclusive tau decay results that rely on Finite Energy Sum Rule techniques and assumptions. Further investigation of that approach seems to be warranted.

Employing the values of V_{ud} and V_{us} from Eq. (4) and Eq. (16), respectively, leads to the unitarity consistency check

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9995(4)(3). \tag{19}$$

where the first error is the uncertainty from $|V_{ud}|^2$ and the second error is the uncertainty from $|V_{us}|^2$.

CKM Unitarity Constraints

The current good experimental agreement with unitarity, $|V_{ud}|^2+|V_{us}|^2+|V_{ub}|^2 = 0.9995(5)$, provides strong confirmation of Standard Model radiative corrections (which range between 3-4% depending on the nucleus used) at better than the 50 sigma level [48]. In addition, it implies constraints on "New Physics" effects at both the tree and quantum loop levels. Those effects could be in the form of contributions to nuclear beta decays, K decays and/or muon decays, with the last of these providing normalization via the muon lifetime [49], which is used to obtain the Fermi constant, $G_\mu = 1.1663787(6) \times 10^{-5}\text{GeV}^{-2}$.

In the following sections, we illustrate the implications of CKM unitarity for (1) exotic muon decays [50](beyond ordinary muon decay $\mu^+ \to e^+\nu_e\bar{\nu}_\mu$) and (2) new heavy quark mixing V_{uD} [51]. Other examples in the literature [52,53] include Z_χ boson quantum loop effects, supersymmetry, leptoquarks, compositeness etc.

Exotic Muon Decays

If additional lepton flavor violating decays such as $\mu^+ \to e^+\bar{\nu}_e\nu_\mu$ (wrong neutrinos) occur, they would cause confusion in searches for neutrino oscillations at, for example, muon storage rings/neutrino factories or other neutrino sources from muon decays. Calling the rate for all such decays Γ(exotic μ decays), they should be subtracted before the extraction of G_μ and normalization of the CKM matrix. Since that is not done and unitarity works, one has (at one-sided 95% CL)

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - BR(\text{exotic } \mu \text{ decays}) \geq 0.9987 \tag{20}$$

or

$$BR(\text{exotic } \mu \text{ decays}) \leq 0.0013 \ . \tag{21}$$

This bound is a factor of 10 better than the direct experimental bound on $\mu^+ \to e^+\bar{\nu}_e\nu_\mu$.

New Heavy Quark Mixing

Heavy D quarks naturally occur in fourth quark generation models and some heavy quark "new physics" scenarios such as E_6 grand unification. Their mixing with ordinary quarks gives rise to V_{ud} which is constrained by unitarity (one sided 95% CL)

$$\begin{aligned} |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 - |V_{uD}|^2 \geq 0.9987 \\ |V_{uD}| \leq 0.04 \ . \end{aligned} \tag{22}$$

A similar constraint applies to heavy neutrino mixing and the couplings $V_{\mu N}$ and V_{eN}.

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