MUON DECAY PARAMETERS

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Introduction: All measurements in direct muon decay, $\mu^- \to e^- + 2$ neutrals, and its inverse, $\nu_\mu + e^- \to \mu^- +$ neutral, are successfully described by the "V-A interaction," which is a particular case of a local, derivative-free, lepton-number-conserving, four-fermion interaction [1]. As shown below, within this framework, the Standard Model assumptions, such as the V-A form and the nature of the neutrals (ν_μ and $\bar{\nu}_e$), and hence the doublet assignments $(\nu_e \;\; e^-)_L$ and $(\nu_\mu \;\; \mu^-)_L$, have been determined from experiments [2,3]. All considerations on muon decay are valid for the leptonic tau decays $\tau \to \ell + \nu_\tau + \bar{\nu}_e$ with the replacements $m_\mu \to m_\tau$, $m_e \to m_\ell$.

Parameters: The differential decay probability to obtain an $e^\pm$ with (reduced) energy between x and $x + \mathrm{d}x$, emitted in the direction $\widehat{\boldsymbol{x}}_3$ at an angle between ϑ and $\vartheta + \mathrm{d}\vartheta$ with respect to the muon polarization vector $\boldsymbol{P}_\mu$, and with its spin parallel to the arbitrary direction $\widehat{\boldsymbol{\zeta}}$, neglecting radiative corrections, is given by

$$\begin{aligned}\frac{\mathrm{d}^2\Gamma}{\mathrm{d}x\ \mathrm{d}\cos\vartheta} =& \frac{m_\mu}{4\pi^3}\, W_{e\mu}^4\ G_\mathrm{F}^2\ \sqrt{x^2 - x_0^2} \\ & \times \left(F_\mathrm{IS}(x) \pm P_\mu \cos\vartheta\ F_\mathrm{AS}(x)\right) \\ & \times \left[1 + \widehat{\boldsymbol{\zeta}} \cdot \boldsymbol{P}_e(x,\vartheta)\right] \ . \end{aligned} \tag{1}$$

Here, $W_{e\mu} = \max(E_e) = (m_\mu^2 + m_e^2)/2m_\mu$ is the maximum $e^\pm$ energy, $x = E_e/W_{e\mu}$ is the reduced energy, $x_0 = m_e/W_{e\mu} = 9.67 \times 10^{-3}$, and $P_\mu = |\boldsymbol{P}_\mu|$ is the degree of muon polarization. $\widehat{\boldsymbol{\zeta}}$ is the direction in which a perfect polarization-sensitive electron detector is most sensitive. The isotropic part of the spectrum, $F_\mathrm{IS}(x)$, the anisotropic part $F_\mathrm{AS}(x)$, and the electron polarization, $\boldsymbol{P}_e(x,\vartheta)$, may be parametrized by the Michel parameter ρ [1], by η [4], by ξ and δ [5,6], *etc.* These are bilinear combinations of the coupling constants $g^\gamma_{\varepsilon\mu}$, which occur in the matrix element (given below).

If the masses of the neutrinos as well as x_0^2 are neglected, the energy and angular distribution of the electron in the rest frame of a muon ($\mu^{\pm}$) measured by a polarization insensitive detector, is given by

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}x\ \mathrm{d}\cos\vartheta} \sim x^2 \cdot \left\{ 3(1-x) + \frac{2\rho}{3}(4x-3) + 3\eta\ x_0(1-x)/x \right.$$
$$\left. \pm\ P_\mu \cdot \xi \cdot \cos\vartheta \left[1 - x + \frac{2\delta}{3}(4x-3) \right] \right\} \ . \qquad (2)$$

Here, ϑ is the angle between the electron momentum and the muon spin, and $x \equiv 2E_e/m_\mu$. For the Standard Model coupling, we obtain $\rho = \xi\delta = 3/4$, $\xi = 1$, $\eta = 0$ and the differential decay rate is

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}x\ \ \mathrm{d}\cos\vartheta} \ = \ \frac{G_{\mathrm{F}}^2 m_\mu^5}{192\pi^3}\ \left[3 - 2x \pm P_\mu \cos\vartheta(2x-1)\right]\ x^2 \quad . \qquad (3)$$

The coefficient in front of the square bracket is the total decay rate.

If only the neutrino masses are neglected, and if the $e^{\pm}$ polarization is detected, then the functions in Eq. (1) become

$$\begin{aligned} F_{\mathrm{IS}}(x) &= x(1-x) + \tfrac{2}{9}\ \rho(4x^2 - 3x - x_0^2) + \eta \cdot x_0(1-x) \\ F_{\mathrm{AS}}(x) &= \tfrac{1}{3}\xi\ \sqrt{x^2 - x_0^2} \\ &\qquad \times \left[1 - x + \tfrac{2}{3}\delta(4x - 3 + (\sqrt{1-x_0^2} - 1))\right] \\ \boldsymbol{P}_e(x,\vartheta) &= P_{\mathrm{T}_1} \cdot \widehat{\boldsymbol{x}}_1 + P_{\mathrm{T}_2} \cdot \widehat{\boldsymbol{x}}_2 + P_L \cdot \widehat{\boldsymbol{x}}_3 \ . \end{aligned} \qquad (4)$$

Here $\widehat{\boldsymbol{x}}_1$, $\widehat{\boldsymbol{x}}_2$, and $\widehat{\boldsymbol{x}}_3$ are orthogonal unit vectors defined as follows:

$\widehat{\boldsymbol{x}}_3$	is along the e momentum $\boldsymbol{p}_e$
$\dfrac{\widehat{\boldsymbol{x}}_3 \times \boldsymbol{P}_\mu}{\lvert\widehat{\boldsymbol{x}}_2 \times \boldsymbol{P}_\mu\rvert} = \widehat{\boldsymbol{x}}_2$	is transverse to $\boldsymbol{p}_e$ and perpendicular to the "decay plane"
$\widehat{\boldsymbol{x}}_2 \times \widehat{\boldsymbol{x}}_3 = \widehat{\boldsymbol{x}}_1$	is transverse to the $\boldsymbol{p}_e$ and in the "decay plane."

The components of $\boldsymbol{P}_e$ then are given by

$$
\begin{aligned}
P_{\mathrm{T}_1}(x,\vartheta) &= P_\mu \sin\vartheta \cdot F_{\mathrm{T}_1}(x) / \left(F_{\mathrm{IS}}(x) \pm P_\mu \cos\vartheta \cdot F_{\mathrm{AS}}(x)\right) \\
P_{\mathrm{T}_2}(x,\vartheta) &= P_\mu \sin\vartheta \cdot F_{\mathrm{T}_2}(x) / \left(F_{\mathrm{IS}}(x) \pm P_\mu \cos\vartheta \cdot F_{\mathrm{AS}}(x)\right) \\
P_L(x,\vartheta) &= \Big(\pm F_{\mathrm{IP}}(x) + P_\mu \cos\vartheta \\
&\qquad \times F_{\mathrm{AP}}(x)\Big) / \left(F_{\mathrm{IS}}(x) \pm P_\mu \cos\vartheta \cdot F_{\mathrm{AS}}(x)\right) \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
F_{\mathrm{T}_1}(x) &= \tfrac{1}{12}\Big\{-2\Big[\xi'' + 12(\rho - \tfrac{3}{4})\Big](1-x)x_0 \\
&\qquad -3\eta(x^2 - x_0^2) + \eta''(-3x^2 + 4x - x_0^2)\Big\} \\
F_{\mathrm{T}_2}(x) &= \tfrac{1}{3}\sqrt{x^2 - x_0^2}\Big\{3\frac{\alpha'}{A}(1-x) + 2\frac{\beta'}{A}\sqrt{1 - x_0^2}\Big\} \\
F_{\mathrm{IP}}(x) &= \tfrac{1}{54}\sqrt{x^2 - x_0^2}\Big\{9\xi'\left(-2x + 2 + \sqrt{1 - x_0^2}\right) \\
&\qquad + 4\xi(\delta - \tfrac{3}{4})(4x - 4 + \sqrt{1 - x_0^2})\Big\} \\
F_{\mathrm{AP}}(x) &= \tfrac{1}{6}\Big\{\xi''(2x^2 - x - x_0^2) + 4(\rho - \tfrac{3}{4})\left(4x^2 - 3x - x_0^2\right) \\
&\qquad + 2\eta''(1-x)x_0\Big\} \ . \qquad (5)
\end{aligned}
$$

For the experimental values of the parameters ρ, ξ, ξ', ξ'', δ, η, η'', α/A, β/A, α'/A, β'/A, which are not all independent, see the Data Listings below. Experiments in the past have also been analyzed using the parameters a, b, c, a', b', c', α/A, β/A, α'/A, β'/A (and $\eta = (\alpha - 2\beta)/2A$), as defined by Kinoshita and Sirlin [5,6]. They serve as a model-independent summary of all possible measurements on the decay electron (see Listings below). The relations between the two sets of parameters are

$$
\begin{aligned}
\rho - \tfrac{3}{4} &= \tfrac{3}{4}(-a + 2c)/A \ , \\
\eta &= (\alpha - 2\beta)/A \ , \\
\eta'' &= (3\alpha + 2\beta)/A \ , \\
\delta - \tfrac{3}{4} &= \tfrac{9}{4} \cdot \frac{(a' - 2c')/A}{1 - [a + 3a' + 4(b + b') + 6c - 14c']/A} \ , \\
1 - \xi\frac{\delta}{\rho} &= 4\,\frac{[(b + b') + 2(c - c')]/A}{1 - (a - 2c)/A} \ , \\
1 - \xi' &= [(a + a') + 4(b + b') + 6(c + c')]/A \ , \\
1 - \xi'' &= (-2a + 20c)/A \ ,
\end{aligned}
$$

where

$$A \;=\; a \;+\; 4b \;+\; 6c\;. \tag{6}$$

The differential decay probability to obtain a *left-handed* ν_e with (reduced) energy between y and $y+dy$, neglecting radiative corrections as well as the masses of the electron and of the neutrinos, is given by [7]

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}y} \;=\; \frac{m_\mu^5\; G_F^2}{16\pi^3} \;\cdot\; Q_L^{\nu_e} \;\cdot y^2 \Big\{(1-y) - \omega_L \cdot (y - \tfrac{3}{4})\Big\}\;. \tag{7}$$

Here, $y = 2\ E_{\nu_e}/m_\mu$. $Q_L^{\nu_e}$ and ω_L are parameters. ω_L is the neutrino analog of the spectral shape parameter ρ of Michel. Since in the Standard Model, $Q_L^{\nu_e} = 1,\ \omega_L = 0$, the measurement of $d\Gamma/dy$ has allowed a null-test of the Standard Model (see Listings below).

Matrix element: All results in direct muon decay (energy spectra of the electron and of the neutrinos, polarizations, and angular distributions), and in inverse muon decay (the reaction cross section) at energies well below $m_W c^2$, may be parametrized in terms of amplitudes $g^\gamma_{\varepsilon\mu}$ and the Fermi coupling constant G_F, using the matrix element

$$\frac{4G_F}{\sqrt{2}} \sum_{\substack{\gamma=S,V,T\\ \varepsilon,\mu=R,L}} g^\gamma_{\varepsilon\mu} \langle \bar{e}_\varepsilon | \Gamma^\gamma | (\nu_e)_n \rangle \langle (\bar{\nu}_\mu)_m | \Gamma_\gamma | \mu_\mu \rangle. \tag{8}$$

We use the notation of Fetscher *et al.* [2], who in turn use the sign conventions and definitions of Scheck [8]. Here, $\gamma = S, V, T$ indicates a scalar, vector, or tensor interaction; and $\varepsilon, \mu = R, L$ indicate a right- or left-handed chirality of the electron or muon. The chiralities n and m of the ν_e and $\bar{\nu}_\mu$ are then determined by the values of γ, ε, and μ. The particles are represented by fields of definite chirality [9].

As shown by Langacker and London [10], explicit lepton-number nonconservation still leads to a matrix element equivalent to Eq. (8). They conclude that it is not possible, even in principle, to test lepton-number conservation in (leptonic) muon decay if the final neutrinos are massless and are not observed.

The ten complex amplitudes $g^\gamma_{\varepsilon\mu}$ (g^T_{RR} and g^T_{LL} are identically zero) and G_F constitute 19 independent (real) parameters to be determined by experiment. The Standard Model interaction corresponds to one single amplitude g^V_{LL} being unity and all the others being zero.

The (direct) muon decay experiments are compatible with an arbitrary mix of the scalar and vector amplitudes g^S_{LL} and g^V_{LL} – in the extreme even with purely scalar $g^S_{LL} = 2$, $g^V_{LL} = 0$. The decision in favour of the Standard Model comes from the quantitative observation of inverse muon decay, which would be forbidden for pure g^S_{LL} [2].

Experimental determination of V−A: In order to determine the amplitudes $g^\gamma_{\varepsilon\mu}$ uniquely from experiment, the following set of equations, where the left-hand sides represent experimental results, has to be solved.

$$
\begin{aligned}
a &= 16(|g^V_{RL}|^2 + |g^V_{LR}|^2) + |g^S_{RL} + 6g^T_{RL}|^2 + |g^S_{LR} + 6g^T_{LR}|^2 \\
a' &= 16(|g^V_{RL}|^2 - |g^V_{LR}|^2) + |g^S_{RL} + 6g^T_{RL}|^2 - |g^S_{LR} + 6g^T_{LR}|^2 \\
\alpha &= 8\mathrm{Re}\left\{g^V_{RL}(g^{S*}_{LR} + 6g^{T*}_{LR}) + g^V_{LR}(g^{S*}_{RL} + 6g^{T*}_{RL})\right\} \\
\alpha' &= 8\mathrm{Im}\left\{g^V_{LR}(g^{S*}_{RL} + 6g^{T*}_{RL}) - g^V_{RL}(g^{S*}_{LR} + 6g^{T*}_{LR})\right\} \\
b &= 4(|g^V_{RR}|^2 + |g^V_{LL}|^2) + |g^S_{RR}|^2 + |g^S_{LL}|^2 \\
b' &= 4(|g^V_{RR}|^2 - |g^V_{LL}|^2) + |g^S_{RR}|^2 - |g^S_{LL}|^2 \\
\beta &= -4\mathrm{Re}\left\{g^V_{RR}g^{S*}_{LL} + g^V_{LL}g^{S*}_{RR}\right\} \\
\beta' &= 4\mathrm{Im}\left\{g^V_{RR}g^{S*}_{LL} - g^V_{LL}g^{S*}_{RR}\right\} \\
c &= \tfrac{1}{2}\left\{|g^S_{RL} - 2g^T_{RL}|^2 + |g^S_{LR} - 2g^T_{LR}|^2\right\} \\
c' &= \tfrac{1}{2}\left\{|g^S_{RL} - 2g^T_{RL}|^2 - |g^S_{LR} - 2g^T_{LR}|^2\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
Q^{\nu_e}_L &= 1 - \left\{\tfrac{1}{4}|g^S_{LR}|^2 + \tfrac{1}{4}|g^S_{LL}|^2 + |g^V_{RR}|^2 + |g^V_{RL}|^2 + 3|g^T_{LR}|^2\right\} \\
\omega_L &= \frac{3}{4}\ \frac{\{|g^S_{RR}|^2 + 4|g^V_{LR}|^2 + |g^S_{RL} + 2g^T_{RL}|^2\}}{|g^S_{RL}|^2 + |g^S_{RR}|^2 + 4|g^V_{LL}|^2 + 4|g^V_{LR}|^2 + 12|g^T_{RL}|^2\}}\ .
\end{aligned}
$$

It has been noted earlier by C. Jarlskog [11], that certain experiments observing the decay electron are especially informative if they yield the V-A values. The complete solution is now found as follows. Fetscher *et al.* [2] introduced four probabilities

$Q_{\varepsilon\mu}(\varepsilon, \mu = R, L)$ for the decay of a μ-handed muon into an ε-handed electron, and showed that there exist upper bounds on Q_{RR}, Q_{LR}, and Q_{RL}, and a lower bound on Q_{LL}. These probabilities are given in terms of the $g^{\gamma}_{\varepsilon\mu}$'s by

$$Q_{\varepsilon\mu} = \frac{1}{4}|g^S_{\varepsilon\mu}|^2 + |g^V_{\varepsilon\mu}|^2 + 3(1-\delta_{\varepsilon\mu})|g^T_{\varepsilon\mu}|^2 \ , \qquad (9)$$

where $\delta_{\varepsilon\mu} = 1$ for $\varepsilon = \mu$, and $\delta_{\varepsilon\mu} = 0$ for $\varepsilon \neq \mu$. They are related to the parameters $a, \ b, \ c, \ a', b'$, and c' by

$$\begin{aligned} Q_{RR} &= 2(b + b')/A \ , \\ Q_{LR} &= [(a - a') + 6(c - c')]/2A \ , \\ Q_{RL} &= [(a + a') + 6(c + c')]/2A \ , \\ Q_{LL} &= 2(b - b')/A \ , \end{aligned} \qquad (10)$$

with $A = 16$. In the Standard Model, $Q_{LL} = 1$ and the others are zero.

Since the upper bounds on Q_{RR}, Q_{LR}, and Q_{RL} are found to be small, and since the helicity of the ν_μ in pion decay is known from experiment [12,13] to very high precision to be -1 [14], the cross section S of *inverse* muon decay, normalized to the V-A value, yields [2]

$$|g^S_{LL}|^2 \leq 4(1 - S) \qquad (11)$$

and

$$|g^V_{LL}|^2 \ = \ S \ . \qquad (12)$$

Thus the Standard Model assumption of a pure V-A leptonic charged weak interaction of e and μ is derived (within errors) from experiments at energies far below mass of the $W^\pm$: Eq. (12) gives a lower limit for V-A, and Eqs. (9) and (11) give upper limits for the other four-fermion interactions. The existence of such upper limits may also be seen from $Q_{RR} + Q_{RL} = (1 - \xi')/2$ and $Q_{RR} + Q_{LR} = \frac{1}{2}(1 + \xi/3 - 16 \ \xi\delta/9)$. Table 1 gives the current experimental limits on the magnitudes of the $g^{\gamma}_{\varepsilon\mu}$'s. More stringent limits on the six coupling constants g^S_{LR}, g^V_{LR}, g^T_{LR}, g^S_{RL}, g^V_{RL}, and g^T_{RL} have been derived from upper limits on the neutrino mass [18]. Limits on the "charge retention" coordinates, as used in the older literature (*e.g.*, Ref. 19), are given by Burkard *et al.* [20].

Table 1. Coupling constants $g^{\gamma}_{\varepsilon\mu}$ and some combinations of them. Ninety-percent confidence level experimental limits. The limits on $|g^S_{LL}|$ and $|g^V_{LL}|$ are from Ref. 15, and the others from a general analysis of muon decay measurements. Top three rows: Ref. 22, fourth row: Ref. 16, next three rows: Ref. 17, last row: Ref. 21. The experimental uncertainty on the muon polarization in pion decay is included. Note that, by definition, $|g^S_{\varepsilon\mu}| \leq 2$, $|g^V_{\varepsilon\mu}| \leq 1$ and $|g^T_{\varepsilon\mu}| \leq 1/\sqrt{3}$.

$\lvert g^S_{RR}\rvert < 0.035$	$\lvert g^V_{RR}\rvert < 0.017$	$\lvert g^T_{RR}\rvert \equiv 0$
$\lvert g^S_{LR}\rvert < 0.050$	$\lvert g^V_{LR}\rvert < 0.023$	$\lvert g^T_{LR}\rvert < 0.015$
$\lvert g^S_{RL}\rvert < 0.420$	$\lvert g^V_{RL}\rvert < 0.105$	$\lvert g^T_{RL}\rvert < 0.105$
$\lvert g^S_{LL}\rvert < 0.550$	$\lvert g^V_{LL}\rvert > 0.960$	$\lvert g^T_{LL}\rvert \equiv 0$
$\lvert g^S_{LR} + 6g^T_{LR}\rvert < 0.143$	$\lvert g^S_{RL} + 6g^T_{RL}\rvert < 0.418$	
$\lvert g^S_{LR} + 2g^T_{LR}\rvert < 0.108$	$\lvert g^S_{RL} + 2g^T_{RL}\rvert < 0.417$	
$\lvert g^S_{LR} - 2g^T_{LR}\rvert < 0.070$	$\lvert g^S_{RL} - 2g^T_{RL}\rvert < 0.418$	
$Q_{RR} + Q_{LR} < 8.2 \times 10^{-4}$		

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