

7. ELECTROMAGNETIC RELATIONS

Revised September 2005 by H.G. Spieler (LBNL).

Quantity	Gaussian CGS	SI
Conversion factors:		
Charge:	$2.997\,924\,58 \times 10^9$ esu	$= 1\text{ C} = 1\text{ A s}$
Potential:	$(1/299.792\,458)$ statvolt (ergs/esu)	$= 1\text{ V} = 1\text{ J C}^{-1}$
Magnetic field:	10^4 gauss = 10^4 dyne/esu	$= 1\text{ T} = 1\text{ N A}^{-1}\text{m}^{-1}$
	$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
	$\nabla \cdot \mathbf{D} = 4\pi\rho$ $\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c} \mathbf{J}$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0$	$\nabla \cdot \mathbf{D} = \rho$ $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}$ $\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$
Constitutive relations:	$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$, $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$	$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$, $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$
Linear media:	$\mathbf{D} = \epsilon\mathbf{E}$, $\mathbf{H} = \mathbf{B}/\mu$ 1 1	$\mathbf{D} = \epsilon\mathbf{E}$, $\mathbf{H} = \mathbf{B}/\mu$ $\epsilon_0 = 8.854\,187 \dots \times 10^{-12}$ F m ⁻¹ $\mu_0 = 4\pi \times 10^{-7}$ N A ⁻²
	$\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$ $\mathbf{B} = \nabla \times \mathbf{A}$	$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ $\mathbf{B} = \nabla \times \mathbf{A}$
	$V = \sum_{\text{charges}} \frac{q_i}{r_i} = \int \frac{\rho(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d^3x'$ $\mathbf{A} = \frac{1}{c} \oint \frac{I d\boldsymbol{\ell}}{ \mathbf{r} - \mathbf{r}' } = \frac{1}{c} \int \frac{\mathbf{J}(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d^3x'$	$V = \frac{1}{4\pi\epsilon_0} \sum_{\text{charges}} \frac{q_i}{r_i} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d^3x'$ $\mathbf{A} = \frac{\mu_0}{4\pi} \oint \frac{I d\boldsymbol{\ell}}{ \mathbf{r} - \mathbf{r}' } = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d^3x'$
	$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}$ $\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \frac{1}{c}\mathbf{v} \times \mathbf{B})$ $\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$ $\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{1}{c}\mathbf{v} \times \mathbf{E})$	$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}$ $\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B})$ $\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$ $\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E})$
	$\frac{1}{4\pi\epsilon_0} = c^2 \times 10^{-7} \text{ N A}^{-2} = 8.987\,55 \dots \times 10^9 \text{ m F}^{-1}$; $\frac{\mu_0}{4\pi} = 10^{-7} \text{ N A}^{-2}$; $c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 2.997\,924\,58 \times 10^8 \text{ m s}^{-1}$	

7.1. Impedances (SI units)

ρ = resistivity at room temperature in $10^{-8} \Omega \text{ m}$:
 ~ 1.7 for Cu ~ 5.5 for W
 ~ 2.4 for Au ~ 73 for SS 304
 ~ 2.8 for Al ~ 100 for Nichrome
 (Al alloys may have double the Al value.)

For alternating currents, instantaneous current I , voltage V , angular frequency ω :

$$V = V_0 e^{j\omega t} = ZI. \quad (7.1)$$

Impedance of self-inductance L : $Z = j\omega L$.

Impedance of capacitance C : $Z = 1/j\omega C$.

Impedance of free space: $Z = \sqrt{\mu_0/\epsilon_0} = 376.7 \Omega$.

High-frequency surface impedance of a good conductor:

$$Z = \frac{(1+j)\rho}{\delta}, \quad \text{where } \delta = \text{skin depth}; \quad (7.2)$$

$$\delta = \sqrt{\frac{\rho}{\pi\nu\mu}} \approx \frac{6.6 \text{ cm}}{\sqrt{\nu \text{ (Hz)}}} \quad \text{for Cu}. \quad (7.3)$$

7.2. Capacitors, inductors, and transmission Lines

The capacitance between two parallel plates of area A spaced by the distance d and enclosing a medium with the dielectric constant ϵ is

$$C = K\epsilon A/d, \quad (7.4)$$

where the correction factor K depends on the extent of the fringing field. If the dielectric fills the capacitor volume without extending beyond the electrodes, the correction factor $K \approx 0.8$ for capacitors of typical geometry.

The inductance at high frequencies of a straight wire whose length ℓ is much greater than the wire diameter d is

$$L \approx 2.0 \left[\frac{\text{nH}}{\text{cm}} \right] \cdot \ell \left(\ln \left(\frac{4\ell}{d} \right) - 1 \right). \quad (7.5)$$

For very short wires, representative of vias in a printed circuit board, the inductance is

$$L(\text{in nH}) \approx \ell/d. \quad (7.6)$$

A transmission line is a pair of conductors with inductance L and capacitance C . The characteristic impedance $Z = \sqrt{L/C}$ and the phase velocity $v_p = 1/\sqrt{LC} = 1/\sqrt{\mu\epsilon}$, which decreases with the inverse square root of the dielectric constant of the medium. Typical coaxial and ribbon cables have a propagation delay of about 5 ns/cm. The impedance of a coaxial cable with outer diameter D and inner diameter d is

$$Z = 60 \Omega \cdot \frac{1}{\sqrt{\epsilon_r}} \ln \frac{D}{d}, \quad (7.7)$$

where the relative dielectric constant $\epsilon_r = \epsilon/\epsilon_0$. A pair of parallel wires of diameter d and spacing $a > 2.5d$ has the impedance

$$Z = 120 \Omega \cdot \frac{1}{\sqrt{\epsilon_r}} \ln \frac{2a}{d}. \quad (7.8)$$

This yields the impedance of a wire at a spacing h above a ground plane,

$$Z = 60 \Omega \cdot \frac{1}{\sqrt{\epsilon_r}} \ln \frac{4h}{d}. \quad (7.9)$$

A common configuration utilizes a thin rectangular conductor above a ground plane with an intermediate dielectric (microstrip). Detailed calculations for this and other transmission line configurations are given by Gunston.*

7.3. Synchrotron radiation (CGS units)

For a particle of charge e , velocity $v = \beta c$, and energy $E = \gamma mc^2$, traveling in a circular orbit of radius R , the classical energy loss per revolution δE is

$$\delta E = \frac{4\pi}{3} \frac{e^2}{R} \beta^3 \gamma^4. \quad (7.10)$$

For high-energy electrons or positrons ($\beta \approx 1$), this becomes

$$\delta E \text{ (in MeV)} \approx 0.0885 [E(\text{in GeV})]^4/R(\text{in m}). \quad (7.11)$$

For $\gamma \gg 1$, the energy radiated per revolution into the photon energy interval $d(\hbar\omega)$ is

$$dI = \frac{8\pi}{9} \alpha \gamma F(\omega/\omega_c) d(\hbar\omega), \quad (7.12)$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant and

$$\omega_c = \frac{3\gamma^3 c}{2R} \quad (7.13)$$

is the critical frequency. The normalized function $F(y)$ is

$$F(y) = \frac{9}{8\pi} \sqrt{3} y \int_y^\infty K_{5/3}(x) dx, \quad (7.14)$$

where $K_{5/3}(x)$ is a modified Bessel function of the third kind. For electrons or positrons,

$$\hbar\omega_c \text{ (in keV)} \approx 2.22 [E(\text{in GeV})]^3/R(\text{in m}). \quad (7.15)$$

Fig. 7.1 shows $F(y)$ over the important range of y .

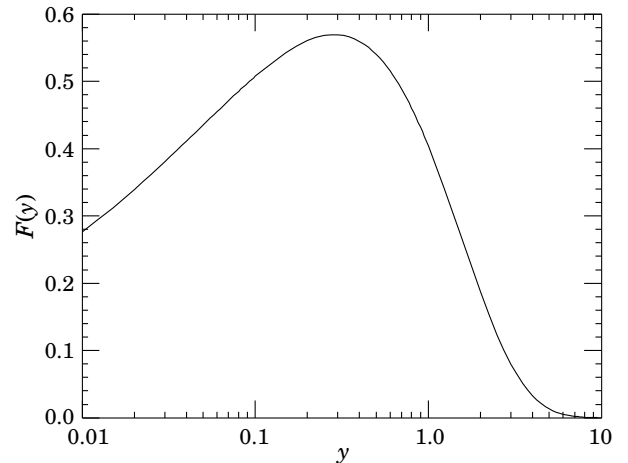


Figure 7.1: The normalized synchrotron radiation spectrum $F(y)$.

For $\gamma \gg 1$ and $\omega \ll \omega_c$,

$$\frac{dI}{d(\hbar\omega)} \approx 3.3\alpha (\omega R/c)^{1/3}, \quad (7.16)$$

whereas for

$$\gamma \gg 1 \text{ and } \omega \gtrsim 3\omega_c,$$

$$\frac{dI}{d(\hbar\omega)} \approx \sqrt{\frac{3\pi}{2}} \alpha \gamma \left(\frac{\omega}{\omega_c} \right)^{1/2} e^{-\omega/\omega_c} \left[1 + \frac{55}{72} \frac{\omega_c}{\omega} + \dots \right]. \quad (7.17)$$

The radiation is confined to angles $\lesssim 1/\gamma$ relative to the instantaneous direction of motion. For $\gamma \gg 1$, where Eq. (7.12) applies, the mean number of photons emitted per revolution is

$$N_\gamma = \frac{5\pi}{\sqrt{3}} \alpha \gamma, \quad (7.18)$$

and the mean energy per photon is

$$\langle \hbar\omega \rangle = \frac{8}{15\sqrt{3}} \hbar\omega_c. \quad (7.19)$$

When $\langle \hbar\omega \rangle \gtrsim O(E)$, quantum corrections are important.

* M.A.R. Gunston. Microwave Transmission Line Data, Noble Publishing Corp., Atlanta (1997) ISBN 1-884932-57-6, TK6565.T73G85.

See J.D. Jackson, *Classical Electrodynamics*, 3rd edition (John Wiley & Sons, New York, 1998) for more formulae and details. (Note that earlier editions had ω_c twice as large as Eq. (7.13).