16. Heavy-Quark and Soft-Collinear Effective Theory

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16.1. Effective Field Theories

Quantum field theories represent the most precise computational tool for describing physics at the highest energies. One of their characteristic features is that they almost inevitably involve multiple length scales. When trying to determine the value of an observable, quantum field theory demands that all possible virtual states and hence all particles be included in the calculation. Since these particles have widely different masses, the final prediction is sensitive to many scales. This fact represents a formidable challenge from a practical point of view. No realistic quantum field theories can be solved exactly, so that one has to resort to approximation schemes; these, however, are typically most straightforward when only a single scale is involved at a time.

Effective field theories (EFTs) provide a general theoretical framework to deal with the multi-scale problems of realistic quantum field theories. This framework aims to reduce such problems to a combination of separate and simpler single-scale problems; simultaneously, however, it provides an organizational scheme whereby the other scales are not omitted but allowed to play their role in a separate step of the computation. The philosophy and basic principles of this approach are very generic, and correspondingly EFTs represent a widely used method in many different areas of high-energy physics, from the low energy scales of atomic and nuclear physics to the high energy scales of (partly yet unknown) elementary particle physics. EFTs can play a role both within analytic perturbative computations and in the context of non-perturbative numerical simulations; see [1–3] for some early references. One of the simplest applications of EFTs to particle physics is to describe an underlying theory that is only probed at energy scales $E < \Lambda$. Any particle with mass $m > \Lambda$ cannot be produced as a real state and therefore only leads to short-distance virtual effects. Thus, one can construct an effective theory in which the quantum fluctuations of such heavy particles are "integrated out" from the generating functional integral for Green functions. This results in a simpler theory containing only those degrees of freedom that are relevant to the energy scales under consideration. In fact, the standard model of particle physics itself is widely viewed as an EFT of some yet unknown, more fundamental theory.

The development of any effective theory starts by identifying the degrees of freedom that are relevant to describe the physics at a given energy (or length) scale, and constructing the Lagrangian describing the interactions among these fields. Short-distance quantum fluctuations associated with much smaller length scales are absorbed into the coefficients of the various operators in the effective Lagrangian. These coefficients are determined in a matching procedure, by requiring that the EFT reproduces the matrix elements of the full theory up to power corrections. In many cases the effective Lagrangian exhibits enhanced symmetries compared with the fundamental theory, allowing for simple and sometimes striking predictions relating different observables.

16.2. Heavy-Quark Effective Theory

Heavy-quark systems provide prime examples for applications of the EFT technology, because the hierarchy $m_Q \gg \Lambda_{\rm QCD}$ (with $Q = b, c$) provides a natural separation of scales. Physics at the scale m_Q is of a short-distance nature and can be treated perturbatively, while for heavy-quark systems there is always also some hadronic physics governed by the confinement scale $\Lambda_{\rm QCD}$. Being able to separate the short-distance and long-distance effects associated with these two scales is crucial for any quantitative description in heavy-quark physics. For instance, if the long-distance hadronic matrix elements are obtained from lattice QCD, then it is necessary to analytically compute the short-distance effects, which come from short-wavelength modes that do not fit on present-day lattices. In many other instances, the long-distance hadronic physics can be encoded in a small number of universal parameters.

16.2.1. ***General idea and derivation of the effective Lagrangian*** : The simplest effective theory for heavy-quark systems is the heavy-quark effective theory (HQET) [4–7] (see [8,9] for detailed discussions). It provides a simplified description of the soft interactions of a single heavy quark interacting with soft, light partons. This includes the interactions that bind the heavy quark with other light partons inside heavy mesons (B, B^*, ...) and baryons (Λ_b, Σ_b, ...).

A softly interacting heavy quark is nearly on-shell. Its momentum may be decomposed as $p_Q = m_Q v + k$, where v is the 4-velocity of the hadron containing the heavy quark, and the "residual momentum" $k \sim \Lambda_{\rm QCD}$ results from the soft interactions of the heavy quark with its environment. In the limit $m_Q \gg \Lambda_{\rm QCD}$, the soft interactions do not change the 4-velocity of the heavy quark, which is therefore a conserved quantum number that is often used as a label on the effective heavy-quark fields. A nearly on-shell Dirac spinor has two large and two small components. We define

$$Q(x) = e^{-im_Q v\cdot x}\left[h_v(x) + H_v(x)\right], \tag{16.1}$$

where

$$h_v(x) = e^{im_Q v\cdot x}\,\frac{1+\not v}{2}\,Q(x)\,, \qquad H_v(x) = e^{im_Q v\cdot x}\,\frac{1-\not v}{2}\,Q(x) \tag{16.2}$$

are the large ("upper") and small ("lower") components of the Dirac spinor, respectively. The extraction of the phase factor in Eq. (16.1) implies that the fields h_v and H_v carry the residual momentum k. These fields obey the projection relations $\not v\, h_v = h_v$ and $\not v\, H_v = -H_v$. Inserting these definitions into the Dirac Lagrangian yields

$$\mathcal{L}_Q = \bar h_v\, iv\cdot D\, h_v + \bar H_v(-iv\cdot D - 2m_Q)H_v + \bar h_v\, i\vec{\not D}\, H_v + \bar H_v\, i\vec{\not D}\, h_v\,, \tag{16.3}$$

where $i\vec{D}^\mu = iD^\mu - v^\mu\, iv\cdot D$ is the "spatial" covariant derivative (note that $v^\mu = (1,\vec{0})$ in the heavy-hadron rest frame). The interpretation of Eq. (16.3) is that the field h_v describes a massless fermion, while H_v describes a heavy fermion with mass $2m_Q$. Both modes are coupled to each other via the last two terms. Soft interactions cannot excite the heavy fermion, so we integrate it out from the generating functional of the theory.

The light field which remains describes the fluctuations of the heavy quark about its mass shell. Solving the classical equation of motion for the field H_v yields

$$H_v = \frac{1}{2m_Q + iv\cdot D}\, i\vec{\not{D}}\, h_v = \frac{1}{2m_Q} \sum_{n=0}^{\infty} \left(-\frac{iv\cdot D}{2m_Q} \right)^n i\vec{\not{D}}\, h_v \,, \tag{16.4}$$

which implies $H_v = O(\Lambda_{\rm QCD}/m_Q)\, h_v$, provided the residual momenta are small. The effective Lagrangian of HQET is obtained by inserting this result into Eq. (16.3). At subleading order in $1/m_Q$ one finds

$$\mathcal{L}_{\rm HQET} = \bar{h}_v\, iv\cdot D_s\, h_v + \frac{1}{2m_Q} \left[\bar{h}_v (i\vec{D}_s)^2 h_v + C_{\rm mag}(\mu)\, \frac{g}{2}\, \bar{h}_v\, \sigma_{\mu\nu}\, G_s^{\mu\nu} h_v \right] + \dots \,. \tag{16.5}$$

Note that the covariant derivative $iD_s^\mu = i\partial^\mu + gA_s^\mu$ contains only the soft gluon field. Hard gluons have been integrated out, and their effects are contained in the Wilson coefficients of the various operators in the effective Lagrangian. From the leading operator one derives the Feynman rules of HQET. The new operators entering at subleading order are referred to as the "kinetic energy" and "chromo-magnetic interaction". The kinetic-energy operator corresponds to the first correction term in the Taylor expansion of the relativistic energy $E = m_Q + \vec{p}^{\,2}/2m_Q + \dots$. Lorentz invariance, which is encoded as a reparametrization invariance of the effective Lagrangian [10], ensures that its Wilson coefficient is not renormalized ($C_{\rm kin} \equiv 1$). The coefficient of the chromo-magnetic operator, $C_{\rm mag}(\mu) = 1 + \mathcal{O}(\alpha_s)$, receives corrections starting at one-loop order.

16.2.2. ***Spin-flavor symmetry and applications in spectroscopy*** : The leading term in the HQET Lagrangian exhibits a global spin-flavor symmetry. Its physical meaning is that, in the infinite mass limit, the properties of hadronic systems containing a single heavy quark are insensitive to the spin and flavor of the heavy quark [11,12]. The spin symmetry results from the fact that there appear no Dirac matrices in the leading term in the effective Lagrangian Eq. (16.5), implying that the interactions of the heavy quark with soft gluons leave its spin unchanged. The flavor symmetry arises since the mass of the heavy quark does not appear at leading order. When there are n_Q heavy quarks moving at the same velocity, one can simply extend Eq. (16.5) by summing over n_Q identical terms for heavy-quark fields h_v^i. The result is invariant under rotations in flavor space. When combined with the spin symmetry, the symmetry group becomes promoted to SU($2n_Q$). The flavor symmetry is broken by the operators arising at order $1/m_Q$ and higher. However, at first order only the chromo-magnetic operator breaks the spin symmetry.

The spin-flavor symmetry leads to many interesting relations between the properties of hadrons containing a heavy quark. The most direct consequences concern the spectroscopy of such states [13]. In the heavy-quark limit, the spin of the heavy quark and the total angular momentum j of the light degrees of freedom are separately conserved by the strong interactions. Because of heavy-quark symmetry, the dynamics is independent of the spin and mass of the heavy quark. Hadronic states can thus be classified by the quantum numbers (flavor, spin, parity, etc.) of the light degrees of freedom. The spin symmetry predicts that, for fixed $j \neq 0$, there is a doublet of degenerate states with total spin $J = j \pm 1/2$. The flavor symmetry relates the properties of states with different heavy quark flavor. In the case of the ground-state mesons containing a heavy quark, the light degrees of freedom have the quantum numbers of an antiquark, and the degenerate states are the pseudoscalar ($J = 0$) and vector ($J = 1$) mesons. Their masses are split by hyperfine corrections of order $1/m_Q$, such that one expects $m_{B^*} - m_B = O(1/m_b)$ and $m_{D^*} - m_D = O(1/m_c)$. It follows that $m_{B^*}^2 - m_B^2 \simeq m_{D^*}^2 - m_D^2 \simeq \text{const}$. The data are compatible with this result: $m_{B^*}^2 - m_B^2 \simeq 0.49\,\text{GeV}^2$ and $m_{D^*}^2 - m_D^2 \simeq 0.55\,\text{GeV}^2$.

16.2.3. ***Weak decay form factors*** : Of particular interest are the relations between the weak decay form factors of heavy mesons, which parametrize hadronic matrix elements of currents between two meson states containing a heavy quark. These relations have been derived by Isgur and Wise [12], generalizing ideas developed by Nussinov and Wetzel [14] and Voloshin and Shifman [15]. For the purpose of this discussion, it is convenient to work with a mass-independent normalization of meson states and use velocity rather than momentum variables.

Consider the elastic scattering of a pseudoscalar meson, $P(v) \to P(v')$, induced by an external vector current coupled to the heavy quark contained in P, which acts as a color source moving with the meson's velocity v. The action of the current is to replace instantaneously the color source by one moving at velocity v'. Soft gluons need to be exchanged in order to rearrange the light degrees of freedom and build the final state meson moving at velocity v'. This rearrangement leads to a form factor suppression. The important observation is that, in the $m_Q \to \infty$ limit, the form factor can only depend on the Lorentz boost $\gamma = v \cdot v'$ connecting the rest frames of the initial and final-state mesons (as long as $\gamma = \mathcal{O}(1)$). In the effective theory, which provides the appropriate framework to consider the limit $m_Q \to \infty$ with the quark velocities kept fixed, the hadronic matrix element describing the scattering process can be written as

$$\langle P(v')|\, \bar{h}_{v'} \gamma^\mu h_v \,|P(v)\rangle = \xi(v \cdot v')(v + v')^\mu, \tag{16.6}$$

with a form factor $\xi(v \cdot v')$ that is real and does not depend on m_Q. By flavor symmetry, the form factor remains identical when one replaces the heavy quark Q in one of the meson states by a heavy quark Q' of a different flavor, thereby turning P into another pseudoscalar meson P'. At the same time, the current becomes a flavor-changing vector current. This universal form factor is called the Isgur-Wise function [12]. For equal velocities the vector current $J^\mu = \bar{h}_v \gamma^\mu h_v$ is conserved in the effective theory, irrespective of the flavor of the heavy quarks. The corresponding conserved charges are the generators of the flavor symmetry. It follows that the Isgur-Wise function is normalized at the point of equal velocities: $\xi(1) = 1$. Since $E_{\text{recoil}} = m_{P'}\,(v \cdot v' - 1)$ is the recoil energy of the daughter meson P' in the rest frame of the parent meson P, the point $v \cdot v' = 1$ is referred to as the zero recoil limit. The heavy-quark spin symmetry leads to additional relations among weak decay form factors. It can be used to relate matrix elements involving vector mesons to those involving pseudoscalar mesons, which once again can be described completely in terms of the universal Isgur-Wise function.

These form factor relations imposed by heavy-quark symmetry describe the semileptonic decay processes $\bar{B} \to D\,\ell\,\bar{\nu}$ and $\bar{B} \to D^*\ell\,\bar{\nu}$ in the limit of infinite heavy-quark masses. They are model-independent consequences of QCD. The known normalization of the Isgur-Wise function at zero recoil can be used to obtain a model-independent measurement of the element $|V_{cb}|$ of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. The semileptonic decay $\bar{B} \to D^*\ell\,\bar{\nu}$ is ideally suited for this purpose [16]. Experimentally, this is a particularly clean mode, since the reconstruction of the D^* meson mass provides a powerful rejection against background. From the theoretical point of view, it is ideal since the decay rate at zero recoil is protected by Luke's theorem against first-order power corrections in $1/m_Q$ [17]. This is described in more detail in Section 11 of the PDG Book.

16.2.4. ***Decoupling transformation*** : At leading order in $1/m_Q$, the couplings of soft gluons to heavy quarks in the effective Lagrangian Eq. (16.5) can be removed by the field redefinition $h_v(x) = Y_v(x)\,h_v^{(0)}(x)$, where $Y_v(x)$ denotes a time-like Wilson line along the direction of v, extending from minus infinity to the point x. In terms of the new fields, the HQET Lagrangian becomes

$$\mathcal{L}_{\mathrm{HQET}} = \bar{h}_v^{(0)}\,iv\cdot\partial\,h_v^{(0)} + O(1/m_Q). \tag{16.7}$$

At leading order in $1/m_Q$, this is a free theory as far as the strong interactions of heavy quarks are concerned. However, the theory is nevertheless non-trivial in the presence of external sources. Consider, e.g., the case of a weak-interaction heavy-quark current

$$\bar{h}_{v'}\gamma^\mu(1-\gamma_5)h_v = \bar{h}_{v'}^{(0)}\gamma^\mu(1-\gamma_5)\,Y_{v'}^\dagger Y_v\,h_v^{(0)}, \tag{16.8}$$

where v and v' are the velocities of the heavy mesons containing the heavy quarks. Unless the two velocities are equal, the object $Y_{v'}^\dagger Y_v$ is non-trivial, and hence the soft gluons do not decouple from the heavy quarks inside the current operator. One may interpret $Y_{v'}^\dagger Y_v$ as a Wilson loop with a cusp at the point x, where the two paths parallel to the different velocity vectors intersect. The presence of the cusp leads to non-trivial ultra-violet behavior (for $v \neq v'$), which is described by a cusp anomalous dimension $\Gamma_c(v\cdot v')$ that was calculated at two-loop order in [18]. It coincides with the velocity-dependent anomalous dimension of heavy-quark currents, which was rediscovered later in the context of HQET [19]. The interpretation of heavy quarks as Wilson lines is a useful tool, which was put forward in some of the very first papers on the subject [4]. This technology will be useful in the study of the interactions of heavy quarks with collinear degrees of freedom discussed later in this review.

16.2.5. *Heavy-quark expansion for inclusive decays* : The theoretical description of inclusive decays of hadrons containing a heavy quark exploits two observations [20–24]: bound-state effects related to the initial state can be calculated using the heavy-quark expansion, and the fact that the final state consists of a sum over many hadronic channels eliminates the sensitivity to the properties of individual final-state hadrons. The second feature rests on the hypothesis of quark-hadron duality, i.e. the assumption that decay rates are calculable in QCD after a smearing procedure has been applied [25]. In semileptonic decays, the integration over the lepton spectrum provides a smearing over the invariant hadronic mass of the final state (global duality). For nonleptonic decays, where the total hadronic mass is fixed, the summation over many hadronic final states provides an averaging (local duality).

Using the optical theorem, the inclusive decay width of a hadron H_b containing a b quark can be written in the form

$$\Gamma(H_b) = \frac{1}{M_{H_b}} \,\mathrm{Im}\, \langle H_b|\, i \int \mathrm{d}^4x\, T\left\{\mathcal{H}_{\rm eff}(x), \mathcal{H}_{\rm eff}(0)\right\} |H_b\rangle \,. \tag{16.9}$$

The effective weak Hamiltonian for b-quark decays consists of dimension-6 four-fermion operators and dipole operators [76]. It follows that the leading contributions to the inclusive decay rate in Eq. (16.9) arise from two-loop diagrams. Because of the large mass of the b quark, the momenta flowing through the internal propagators are large. It is thus possible to construct an operator-product expansion (OPE) for the transition operator, in which it is represented as a series of local operators containing two b-quark fields. The operator with the lowest dimension is $\bar{b}b$. The next non-trivial operator has dimension 5 and contains the gluon field. It arises from diagrams in which a soft gluon is emitted from one of the internal lines of the two-loop diagrams. From dimension 6 on, an increasing number of operators appears. For dimensional reasons, the matrix elements of higher-dimensional operators are suppressed by inverse powers of the b-quark mass. Thus, the total inclusive decay rate of a hadron H_b can be written as [21,45]

$$\Gamma(H_b) = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} \left\{ c_3\, \langle \bar{b}b \rangle + c_5\, \frac{\langle \bar{b}\, g\sigma_{\mu\nu} G^{\mu\nu} b\rangle}{m_b^2} + \sum_n c_6^{(n)}\, \frac{\langle O_6^{(n)} \rangle}{m_b^3} + \dots \right\}, \tag{16.10}$$

where the prefactor arises from the loop integrations, c_i are calculable coefficient functions, and $\langle O_i \rangle$ are the (normalized) forward matrix elements between H_b states. These matrix elements can be systematically expanded in powers of $1/m_b$ using HQET. The result is [21,45]

$$\langle \bar{b}b \rangle = 1 - \frac{\mu_\pi^2(H_b) - \mu_G^2(H_b)}{2m_b^2} + \dots, \qquad \frac{\langle \bar{b}\, g\sigma_{\mu\nu} G^{\mu\nu} b\rangle}{m_b^2} = \frac{2\mu_G^2(H_b)}{m_b^2} + \dots, \tag{16.11}$$

where $\mu_\pi^2(H_b)$ and $\mu_G^2(H_b)$ are the matrix elements of the heavy-quark kinetic energy and chromomagnetic interaction inside the hadron H_b, respectively [27]. For the ground-state heavy mesons and baryons, one can extract $\mu_G^2(B) = 3(m_{B^*}^2 - m_B^2)/4 \simeq 0.36\,\mathrm{GeV}^2$ and $\mu_G^2(\Lambda_b) = 0$ from spectroscopy.

From the fully inclusive width Eq. (16.10) one can obtain the lifetime of a heavy hadron via $\tau(H_b) = 1/\Gamma(H_b)$. Due to the universality of the leading term in the heavy-quark expansion, lifetime ratios such as $\tau(B^-)/\tau(\bar{B}^0)$, $\tau(\bar{B}_s^0)/\tau(\bar{B}^0)$, and $\tau(\Lambda_b)/\tau(\bar{B}^0)$ are particularly sensitive to the hadronic parameters determining the power corrections in the expansion. In order to understand these ratios theoretically, it is necessary to include phase-space enhanced power corrections of order $(\Lambda_{\rm QCD}/m_b)^3$ as well as short-distance perturbative effects in the calculation [28,29].

A formula analogous to Eq. (16.10) can be derived for differential distributions in specific inclusive decay processes, assuming that these distributions are integrated over sufficiently large portions of phase space to ensure quark-hadron duality. Important examples are the distributions in lepton energy ($d\Gamma/dE_\ell$) or lepton invariant mass ($d\Gamma/dq^2$), as well as moments of the invariant hadronic mass distribution, in the semileptonic processes $\bar{B} \to X_u\,\ell\,\bar{\nu}$ and $\bar{B} \to X_c\,\ell\,\bar{\nu}$, as well as the photon energy spectrum ($d\Gamma/dE_\gamma$) in the radiative process $\bar{B} \to X_s\gamma$. While the latter process is primarily used to test the Standard Model and search for hints of new physics, an analysis of decay distributions in the semileptonic processes can be employed to perform a global fit determining the CKM matrix elements $|V_{ub}|$ and $|V_{cb}|$ along with heavy-quark parameters such as the masses m_b, m_c and the hadronic parameters $\mu_\pi^2(B)$, $\mu_G^2(B)$. These determinations provide some of the most accurate values for these parameters [30].

16.2.6. ***Shape functions and non-local power corrections*** : In certain regions of phase space, in which the hadronic final state in an inclusive heavy-hadron decay is made up of light energetic partons, the local OPE for inclusive decays must be replaced by a more complicated expansion involving hadronic matrix elements of non-local light-ray operators [50,51]. Prominent examples are the radiative decay $\bar{B} \to X_s\gamma$ for large photon energy E_γ near $m_B/2$, and the semileptonic decay $\bar{B} \to X_u\,\ell\,\bar{\nu}$ at large lepton energy or small hadronic invariant mass. In these cases, the differential decay rates at leading order in the heavy-quark expansion can be written in the factorized form $d\Gamma \propto H\,J \otimes S$ [33], where the hard function H and the jet function J are calculable in perturbation theory. The characteristic scales for these functions are set by m_b and $(m_b\Lambda_{\rm QCD})^{1/2}$, respectively. The soft function

$$S(\omega) = \int \frac{dt}{4\pi}\, e^{-i\omega t}\, \langle \bar{B}(v)|\, \bar{h}_v(tn) Y_n(tn) Y_n^\dagger(0) h_v(0) |\bar{B}(v)\rangle \tag{16.12}$$

is a genuinely non-perturbative object, called the shape function [50,51]. Here Y_n are soft Wilson lines along a light-like direction n aligned with the momentum of the hadronic final-state jet. The jet function and the shape function share a common variable $\omega \sim \Lambda_{\rm QCD}$, and the symbol $\otimes$ denotes a convolution in this variable.

While the hard function is different for the two decays, the jet and soft functions are identical at leading order in $\Lambda_{\rm QCD}/m_Q$. This is particularly important for the soft function. It is this shape function that introduces non-perturbative physics into the theoretical predictions for the cross sections of $\bar{B} \to X_s\gamma$ and $\bar{B} \to X_u\,\ell\bar{\nu}$ in the regions of experimental interest. The fact that both decays depend on the same non-perturbative function made it possible to determine this non-perturbative information from the measured shape of the photon spectrum in $\bar{B} \to X_s\gamma$, allowing for a better understanding of the process used to determine the CKM element $|V_{ub}|$. In higher orders of the heavy-quark expansion, an increasing number of subleading jet and soft functions is required to describe the decay distributions [34]. These have been analyzed in detail at order $1/m_b$ [35–37]. The technology for deriving the corresponding factorization theorems relies on SCET, which is discussed below.

16.3. Soft-Collinear Effective Theory

As discussed in the previous section, soft gluons that bind a heavy quark inside a heavy meson cannot change the virtuality of that heavy quark by a significant amount. The ratio of $\Lambda_{\rm QCD}/m_Q$ provided the expansion parameter in HQET, which is a small parameter since $m_Q \gg \Lambda_{\rm QCD}$. This obviously does not work when considering light quarks. However, if the energy Q of the quarks is large, the ratio $\Lambda_{\rm QCD}/Q$ provides a small parameter which can be used to construct an effective theory. One major difference to HQET is that light energetic quarks cannot only emit soft gluons, but they can also emit collinear gluons (an energetic gluon in the same direction as the original quark), without parametrically changing their virtuality. Thus, to fully reproduce the long-distance physics of energetic quarks requires that one includes their interactions with both soft and collinear particles. The resulting effective theory is therefore called soft-collinear effective theory (SCET) [38–40].

SCET is applicable for processes containing particles with energy much in excess of their mass, and it has therefore a wide range of applications. In this brief review we will outline the main features of this effective theory and mention a few selected applications.

16.3.1. ***General idea of the expansion*** : Consider a quark with energy Q and virtuality $m \ll Q$, moving along the direction $\vec{n}$. It is convenient to parameterize the momentum p_n of this particle in terms of its light-cone components, defined by $(p_n^-, p_n^+, p_n^\perp) = (\bar{n}\cdot p_n, n\cdot p_n, p_n^\perp)$, where $n^\mu = (1,\vec{n})$ and $\bar{n}^\mu = (1,-\vec{n})$ are light-like 4-vectors, and $n\cdot p_\perp = \bar{n}\cdot p_\perp = 0$. A subscript n has been added to the momentum to identify it as a collinear particle in direction n (more precisely, a particle with energy much larger than its virtuality moving along a direction $\vec{n}$). In terms of these light-cone components, the virtuality satisfies $m^2 = p_n^+ p_n^- + p_n^{\perp 2}$. The individual components of the momentum satisfy

$$(p_n^-, p_n^+, p_n^\perp) \sim (Q, m^2/Q, m) \equiv Q(1,\lambda^2,\lambda), \tag{16.13}$$

where $\lambda = m/Q$ is the expansion parameter of SCET.

The virtuality of such an energetic particle remains parametrically unchanged if it interacts with energetic particles in the same direction n, or with soft particles with momentum scaling as

$$(p_s^-, p_s^+, p_s^\perp) \sim Q(\lambda^2,\lambda^2,\lambda^2). \tag{16.14}$$

Thus, it is the interactions of collinear and soft degrees of freedom that give rise to the long-distance physics. SCET, which is constructed to reproduce this long-distance dynamics, is therefore an effective theory describing the interactions of collinear and soft particles.

The above power counting treats the soft momentum to be of order m^2/Q, where m denotes the mass of a collinear system. If the mass of the collinear system is of order $\Lambda_{\rm QCD}$, as would be the case for a single energetic hadron, this power counting is no longer applicable, since $\Lambda_{\rm QCD}$ provides a natural cutoff to QCD and the soft momentum cannot be below this scale. To describe such systems requires a modified version of SCET, called $\rm SCET_{II}$, in which the scaling of the soft modes is $Q(\lambda,\lambda,\lambda)$. In this review we will focus only on SCET with the scaling discussed before, which is sometimes called $\rm SCET_I$.

16.3.2. ***Leading-order Lagrangian*** : The derivation of the SCET Lagrangian follows similar steps as the derivation of the HQET Lagrangian in Section 16.2.1, but care has to be taken to properly account for the interactions of collinear fields with one another. We begin by deriving the Lagrangian for a theory containing only a single type of collinear degrees of freedom. We are interested in the interactions of fermion fields $q_n(x)$ with gluon fields $A_n(x)$, which have collinear momentum in the same light-like direction n. Similar to HQET, one can separate the full QCD field into two components, $q_n(x) = \psi_n(x) + \Xi_n(x)$, where

$$\psi_n(x) = \frac{\not{n}\not{\bar{n}}}{4}\, q_n(x)\,, \qquad \Xi_n(x) = \frac{\not{\bar{n}}\not{n}}{4}\, q_n(x).$$

In terms of these fields, the QCD Lagrangian is

$$\begin{aligned}\mathcal{L}_n = \bar{\psi}_n(x)\frac{\not{\bar{n}}}{2}\, in\cdot D_n\, \psi_n(x) + \bar{\Xi}_n(x)\frac{\not{n}}{2}\, i\bar{n}\cdot D_n\, \Xi_n(x)\\ + \bar{\psi}_n(x) i\not{D}_n^{\perp}\, \Xi_n(x) + \bar{\Xi}_n(x) i\not{D}_n^{\perp}\, \psi_n(x),\end{aligned} \tag{16.15}$$

where we have defined the transverse derivative $D_n^{\perp\,\mu} = D_n^\mu - \dfrac{n^\mu}{2}\bar{n}\cdot D_n - \dfrac{\bar{n}^\mu}{2} n\cdot D_n$. Since $\bar{n}\cdot p_n \gg 1$ the field $\Xi_n(x)$ has no pole in its propagator, similar to the field $H_v(x)$ in Eq. (16.3). It can therefore be integrated out using its equation of motion. Inserting this back into Eq. (16.15), we find

$$\mathcal{L}_n = \bar{\psi}_n(x)\left[in\cdot D_n + i\not{D}_n^{\perp}\frac{1}{i\bar{n}\cdot D_n} i\not{D}_n^{\perp}\right]\frac{\not{\bar{n}}}{2}\,\psi_n(x). \tag{16.16}$$

While this Lagrangian leads to the correct Feynman rules of SCET, there is one feature that warrants extra discussion. In contrast to the Lagrangian of HQET given in Eq. (16.5), where the derivative scales like the residual momentum k of the heavy quark, the derivatives in Eq. (16.16) pick up both the large momentum components of order Q and $Q\lambda$, as well as the residual momentum of order $Q\lambda^2$. One can separate the large and residual momentum components using a procedure similar to the HQET case. Separating the collinear momentum into a "label" and a residual component, $p^\mu = P^\mu + k^\mu$, and performing a phase redefinition on the collinear fields $\psi_n(x) = e^{iP\cdot x}\,\xi_n(x)$, derivatives acting on the fields $\xi_n(x)$ now only pick out the residual momentum. Since the label momentum in SCET is not conserved as in HQET, one defines a label operator $\mathcal{P}^\mu$

acting as $\mathcal{P}^\mu \xi_n(x) = P^\mu \xi_n(x)$ [39], as well as a corresponding covariant label operator $i\mathcal{D}_n^\mu = \mathcal{P}^\mu + gA_n(x)$.

The final step to complete the Lagrangian of SCET is to include the interactions of collinear fields with soft fields. These interactions can be included by adding the soft gluons to the covariant derivatives, while preserving the power counting. This leads to the final SCET Lagrangian [39–41]

$$\mathcal{L}_n = \bar{\xi}_n(x) \left[in \cdot D_n + gn \cdot A_s + i\not{\mathcal{D}}_n^\perp \frac{1}{i\bar{n} \cdot \mathcal{D}_n} i\not{\mathcal{D}}_n^\perp \right] \frac{\not{\bar{n}}}{2} \xi_n(x). \tag{16.17}$$

The leading-order Lagrangian describing collinear fields in different light-like directions is simply given by the sum of the Lagrangians for each direction n separately, i.e. $\mathcal{L} = \sum_n \mathcal{L}_n$. The soft gluons are the same in each individual Lagrangian. An alternative way to understand the separation between large and small momentum components is to derive the Lagrangian of SCET in position space. In this case no label operators are required to describe interactions in SCET, and the dependence on short-distance effects is contained in non-localities at short distances. An important difference between SCET and HQET is that the SCET Lagrangian is not corrected by short distance fluctuations [42].

16.3.3. ***Collinear gauge invariance and Wilson lines*** : An important aspect of SCET is the gauge structure of the theory. Because the effective field operators in SCET describe modes with certain momentum scalings, the effective Lagrangian respects only residual gauge symmetries. One of them satisfies the collinear scaling

$$(\bar{n} \cdot \partial_n, n \cdot \partial_n, \partial_n^\perp) U_n(x) \sim Q(1, \lambda^2, \lambda) U_n(x), \tag{16.18}$$

and one the soft scaling

$$(\bar{n} \cdot \partial_n, n \cdot \partial_n, \partial_n^\perp) U_s(x) \sim Q(\lambda^2, \lambda^2, \lambda^2) U_s(x). \tag{16.19}$$

The fact that collinear fields in different directions do not transform under the same gauge transformations implies that each collinear sector, containing particles with large momenta along a certain direction, has to be separately gauge invariant. This is achieved by the introduction of collinear Wilson lines [39]

$$W_n(x) = P \exp \left[-ig \int_{-\infty}^{0} ds\, \bar{n} \cdot A_n(s\bar{n} + x) \right], \tag{16.20}$$

which transform under collinear gauge transformations according to $W_n \to U_n W_n$. Thus, the combination $\chi_n \equiv W_n^\dagger \psi_n$ is gauge invariant. In a similar manner, one can define the gauge-invariant gluon field $B_n^\mu = g^{-1} W_n^\dagger iD_n^\mu W_n$ [43]. Operators in SCET are typically constructed from such gauge-invariant collinear fields.

16.3.4. ***Decoupling of soft gluons*** : Soft gluons in SCET couple to collinear quarks only through the term $\bar{\xi}_n \, gn \cdot A_s \, \frac{\bar{\not{n}}}{2} \, \xi_n$ in the effective Lagrangian in Eq. (16.17). This coupling is similar to the coupling of soft gluons to heavy quarks in HQET, and soft gluons in SCET can be decoupled from collinear fields in a way similar as explained in Section 16.2.4. Written in terms of the redefined fields

$$\psi_n(x) = Y_n(x)\psi_n^{(0)}(x)\,, \qquad A_n(x) = Y_n(x)A_n^{(0)}(x)Y_n^\dagger(x), \tag{16.21}$$

the soft gluons decouple from the SCET Lagrangian [40]. This fact greatly facilitates proofs of factorization theorems in SCET.

16.3.5. ***Factorization Theorems*** : One of the important applications of SCET is to understand how to factorize cross sections involving energetic particles in different directions into simpler pieces that can either be calculated perturbatively or determined from data. Factorization theorems have been around for much longer than SCET. For a review on the subject, see [44]. However, the effective theory allows for a conceptually simpler understanding of certain classes of factorization theorems [45], since most simplifications happen already at the level of the Lagrangian. The discussion in this section is valid to leading order in the power counting of the effective theory.

As discussed in the previous section, the Lagrangian of SCET does not involve any couplings between collinear degrees of freedom in different light-like directions, or between soft and collinear degrees of freedom after the field redefinition Eq. (16.21) has been performed. An operator describing the scattering and production of collinear partons at short distances can thus be written as

$$\langle O(x)\rangle \simeq C_O(\mu)\,\left\langle \mathcal{C}_{n_a}^{(0)}(x)\mathcal{C}_{n_b}^{(0)}(x)\mathcal{C}_{n_1}^{(0)}(x)\ldots\mathcal{C}_{n_N}^{(0)}(x)[\mathcal{Y}_{n_a}\mathcal{Y}_{n_b}\mathcal{Y}_{n_1}\ldots\mathcal{Y}_{n_N}](x)\right\rangle_\mu . \tag{16.22}$$

Here $\mathcal{C}_n(x)$ denotes a gauge-invariant combination of collinear fields (either quark or gluon fields) in the direction n, and the matching coefficient accounting for short-distance effects is denoted by C_O. The soft Wilson lines can either be in a color triplet or color octet representation, and are collectively denoted by $\mathcal{Y}_n$. Both the matrix elements and the coefficient C_O depend on the renormalization scale μ.

Having defined the operator mediating a given process, one can calculate the cross section by squaring the operator, taking the forward matrix element and integrating over the phase space of all final-state particles. The absence of interactions between collinear degrees of freedom moving along different directions or soft degrees of freedom implies that the forward matrix element of the operator can be factorized as

$$\begin{aligned}\left\langle \mathrm{in}\left|O(x)O^\dagger(0)\right|\mathrm{in}\right\rangle = &\left\langle \mathrm{in}_a\left|\mathcal{C}_{n_a}(x)\mathcal{C}_{n_a}^\dagger(0)\right|\mathrm{in}_a\right\rangle \left\langle \mathrm{in}_b\left|\mathcal{C}_{n_b}(x)\mathcal{C}_{n_b}^\dagger(0)\right|\mathrm{in}_b\right\rangle \\ &\times\left\langle 0\left|\mathcal{C}_{n_1}(x)\mathcal{C}_{n_1}^\dagger(0)\right|0\right\rangle\cdots\left\langle 0\left|\mathcal{C}_{n_N}(x)\mathcal{C}_{n_N}^\dagger(0)\right|0\right\rangle \\ &\times\left\langle 0\left|[\mathcal{Y}_{n_a}\ldots\mathcal{Y}_{n_N}](x)[\mathcal{Y}_{n_a}\ldots\mathcal{Y}_{n_N}]^\dagger(0)\right|0\right\rangle .\end{aligned} \tag{16.23}$$

Thus, the matrix element required for the differential cross section has factorized into a product of simpler structures, each of which can be evaluated separately.

For most applications the matrix elements of incoming collinear fields are non-perturbative objects given in terms of the well-known parton distribution functions, while the matrix elements of outgoing collinear fields are determined by perturbatively calculable jet functions $J_i(\mu)$. Finally, the vacuum matrix element of the soft Wilson lines defines a so-called soft function, commonly denoted by $S(\mu)$. The common dependence on x in the above equation implies that in momentum space the various components of the factorization theorem are convoluted with one another. Deriving this convolution requires a careful treatment of the phase-space integration, in particular treating the large and residual components of each momentum appropriately.

Putting all information together, the differential cross section can be written as

$$\mathrm{d}\sigma \sim H(\mu) \otimes \left[f_{p_1/P}(\mu) f_{p_2/P}(\mu)\right] \otimes [J_1(\mu) \ldots J_N(\mu)] \otimes S(\mu). \tag{16.24}$$

The hard coefficient is equal to the square of the matching coefficient, $H(\mu) = |C_O(\mu)|^2$. It should be mentioned that the most difficult part of traditional factorization proofs involves showing that so-called Glauber gluons do not spoil the above factorization theorem. This question has not yet been fully addressed in the context of SCET.

16.3.6. ***Resummation of large logarithms*** : SCET can be used to sum the large logarithmic terms that arise in perturbative calculations. In general, perturbation theory will generate a logarithmic dependence on any ratio of scales r in a problem, and for processes that involve initial or final states with energy much in excess of their mass there are two powers of logarithms for every power of the strong coupling constant. Thus, for widely separated scales these large logarithms can spoil fixed-order perturbation theory, and a much better convergence is achieved by expanding in α_s, but holding $\alpha_s \log^2 r$ fixed, such that the first term in the new expansion resums powers of $\alpha_s \log^2 r$ to all orders. More precisely, a proper resummation requires to sum logarithms of the form $\alpha_s^n \log^m r$ with $m \leq n+1$ in the logarithm of a cross-section.

The important ingredient in achieving this resummation is the fact that SCET factorizes a given cross section into simpler pieces, as discussed in the previous section. Each of the ingredients of the factorization theorem depends on a single physical scale, and the only dependence on that scale can arise through logarithms of its ratio with the renormalization scale μ. Thus, for each of the components in the factorization theorem one can choose a renormalization scale μ for which the large logarithmic terms are absent.

Of course, the factorization formula requires a common renormalization scale μ in all its components, and one therefore has to use the RG to evolve the various component functions from their preferred scale to the common scale μ. For example, for the hard coefficient $H(\mu)$, the RG equation can be written as

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} H(\mu) = \gamma_H(\mu) H(\mu). \tag{16.25}$$

In general, the anomalous dimension is of the form $\gamma_H(\mu) = c_H\,\Gamma_{\rm cusp}(\alpha_s)\log(Q/\mu) + \gamma(\alpha_s)$, where c_H is a process-dependent coefficient and $\Gamma_{\rm cusp}$ denotes the so-called cusp anomalous dimension [18,46]. The non-cusp part of the anomalous dimension γ is again process dependent. The presence of a logarithm of the hard scale Q in the anomalous dimension is characteristic of Sudakov problems and arises since the perturbative series contains double logarithms of scale ratios. The anomalous dimension γ_H is known at two-loop order for arbitrary n-parton amplitudes containing massless or massive external partons [47–50]. Solving the RG equation yields

$$H(\mu) = U_H(\mu,\mu_h)H(\mu_h)\,, \tag{16.26}$$

which can be used to write the hard function at a scale $\mu_h \sim Q$, where its perturbative expression does not contain any large logarithms, in terms of the common renormalization scale μ. The RG evolution factor $U_H(\mu,\mu_h)$ sums logarithms of the form μ/μ_h. By calculating the anomalous dimension $\gamma_H(\mu)$ to higher and higher orders in perturbation theory, one can resum more and more logarithms in the evolution kernel. The RG equations for the jet and soft functions are more complicated, since they involve convolutions over the relevant momentum variables.

16.3.7. ***Applications*** : Most of the applications of SCET are either in flavor physics, where the decay of a heavy B meson can give rise to energetic light partons, or in collider physics, where the presence of jets naturally leads to collimated sets of energetic particles. For several of these applications alternative approaches existed before the invention of SCET, but the effective theory has opened up alternative ways to understand the physics of these processes. There are, however, many examples for which SCET has allowed new insights that were not available or possible without the effective theory. In particular, it has provided a field-theoretic basis for the QCD factorization approach to exclusive, non-leptonic decays of B mesons [51]. Using SCET methods, proofs of factorization were derived for the color-allowed decay $\bar{B}^0 \to D^+\pi^-$ [52], the color-suppressed decay $\bar{B}^0 \to D^0\pi^0$ [53], and the radiative decay $\bar{B} \to K^*\gamma$ [54]. Further examples are factorization theorems and the resummation of endpoint logarithms for quarkonia production [55], factorization theorems for cross sections defined through jet algorithms [56], the resummation of large logarithmic terms for the thrust [57] and jet broadening [58] distributions in e^+e^- annihilation beyond NLL order, the development of new factorizable observables to veto extra jets [59], all-orders factorization theorems for processes containing electroweak Sudakov logarithms [60], as well as the resummation of threshold (soft gluon) logarithms for several important processes at hadron colliders [61,62]. We describe two of these applications in more detail.

Event-shape distributions, in particular the thrust distribution, have been measured to high accuracy at LEP [63]. Comparing these data to precise theoretical predictions allows for a determination of the strong coupling constant α_s. For small values of $\tau \equiv 1 - T$, the distribution can be factorized into the form [64,65]

$$\frac{1}{Q\sigma_0}\frac{{\rm d}\sigma}{{\rm d}\tau} = H(\mu)\int {\rm d}s \int {\rm d}k\, J(s,\mu)\, S(Q\tau - s/Q - k,\mu). \tag{16.27}$$

Here Q denotes the center-of-mass energy of the collision, σ_0 is the total hadronic cross section, and H, J and S are the hard, jet and soft functions in SCET. Large logarithms of the form $(\alpha_s^n \ln^{2n-1} \tau)/\tau$ become important and have to be resummed. Furthermore, for $\tau \sim \Lambda_{\rm QCD}/Q$ non-perturbative effects in the soft function become important. Using SCET the resummation of these large logarithms has been performed to N^{3}LL, which is two orders beyond what was previously available [57]. The factorization in the effective theory has also allowed to include the non-perturbative physics through a shape function, very similar to the B-physics case discussed above. The known perturbative effects for large values of τ can be included by matching the SCET result to the known two-loop spectrum [66,67]. Comparing the predicted to the measured thrust distribution allows for a precise determination of the strong coupling constant α_s [68].

For quarkonium produced in e^+e^- annihilation or photo-production, large logarithms arise in the region of phase space where the energy of the produced $Q\bar{Q}$ state is close to its maximum value. In this region the quarkonium is predominantly produced in a color-octet configuration and recoils against a collinear hadronic system. Then logarithms of the form $\log(1 - E_\Psi/E_{\rm max})$ as well as non-perturbative effects become important and should be included in attempts to describe the data. While some of these issues had been addressed without the use of an effective theory (see [69] and references therein), a complete treatment of the endpoint region has only been achieved using SCET [55]. It was shown that including both effects consistently, theory and data can be brought into better agreement.

16.4. Open issues and perspectives

HQET has successfully passed many experimental tests, and there are not too many open questions that still need to be addressed. One issue that has not been derived from first principles is quark-hadron duality. The validity of global duality (at energies even lower than those relevant in B decays) has been tested experimentally using high-precision data on semileptonic B decays and on hadronic τ decays, and there has been good agreement between theory and data. However, assigning a theoretical uncertainty to possible duality violations is difficult. Another known issue is the that the measured value of the CKM element $|V_{cb}|$ is different depending of whether one uses inclusive or exclusive B decays to derive it (see the relevant section in the Particle Data Book). Both measurements rely on the heavy-quark limit, and the uncertainties quoted include the effects from power corrections arising from the finite b-quark mass.

SCET, on the other hand, is still an active field of research, and there are several open questions that need to be answered. In this review we have not discussed any issues having to do with SCET$_{\rm II}$, which is the appropriate effective theory describing the interactions of collinear particles with soft particles having momentum scaling as $Q(\lambda, \lambda, \lambda)$. This is important, for example, to describe exclusive decays of B mesons into light, energetic mesons, or in collider-physics applications such as the p_T resummation for Drell-Yan production. There are still open issues in how to properly formulate SCET$_{\rm II}$, which are under active investigation. They include the treatment of endpoint singularities of convolution integrals, double counting between overlapping momentum regions, and the breakdown of the naive factorization of soft and collinear modes due to quantum effects. Glauber gluons are known to affect factorization theorems, but how to properly include them in SCET is still an open question.

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