25. ACCELERATOR PHYSICS OF COLLIDERS

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25.1. Luminosity

The event rate R in a collider is proportional to the interaction cross section σ_{int} and the factor of proportionality is called the *luminosity*:

$$R = \mathscr{L}\sigma_{\text{int}} \ . \tag{25.1}$$

If two bunches containing n_1 and n_2 particles collide with frequency f, the luminosity is

$$\mathscr{L} = f\frac{n_1 n_2}{4\pi\sigma_x\sigma_y} \tag{25.2}$$

where σ_x and σ_y characterize the Gaussian transverse beam profiles in the horizontal (bend) and vertical directions and to simplify the expression it is assumed that the bunches are identical in transverse profile, that the profiles are independent of position along the bunch, and the particle distributions are not altered during collision. Whatever the distribution at the source, by the time the beam reaches high energy, the normal form is a good approximation thanks to the central limit theorem of probability and the diminished importance of space charge effects.

The beam size can be expressed in terms of two quantities, one termed the *transverse emittance*, ϵ, and the other, the *amplitude function*, β. The transverse emittance is a beam quality concept reflecting the process of bunch preparation, extending all the way back to the source for hadrons and, in the case of electrons, mostly dependent on synchrotron radiation. The amplitude function is a beam optics quantity and is determined by the accelerator magnet configuration. When expressed in terms of σ and β the transverse emittance becomes

$$\epsilon = \pi\sigma^2/\beta \ . \tag{25.3}$$

Of particular significance is the value of the amplitude function at the interaction point, β^*. Clearly one wants β^* to be as small as possible; how small depends on the capability of the hardware to make a near-focus at the interaction point.

Eq. (25.2) can now be recast in terms of emittances and amplitude functions as

$$\mathscr{L} = f\frac{n_1 n_2}{4\sqrt{\epsilon_x\,\beta_x^*\,\epsilon_y\,\beta_y^*}} \ . \tag{25.4}$$

Thus, to achieve high luminosity, all one has to do is make high population bunches of low emittance to collide at high frequency at locations where the beam optics provides as low values of the amplitude functions as possible.

25.2. Beam dynamics

Today's operating HEP colliders are all synchrotrons, and the organization of this section reflects that circumstance.

A major concern of beam dynamics is stability: conservation of adequate beam properties over a sufficiently long time scale. Several time scales are involved, and the approximations used in writing the equations of motion reflect the time scale under consideration. For example, when, in Sec. 25.2.1 below, we write the equations for transverse stability no terms associated with phase stability or synchrotron radiation appear; the time scale associated with the last two processes is much longer than that demanded by the need for transverse stability.

25.2.1. *Betatron oscillations*:

Present-day high-energy accelerators employ alternating gradient focussing provided by quadrupole magnetic fields [1], [2]. The equations of motion of a particle undergoing oscillations with respect to the design trajectory are

$$x'' + K_x(s)\, x = 0 \ , \quad y'' + K_y(s)\, y = 0 \ , \tag{25.5}$$

with

$$x' \equiv dx/ds \ , \quad y' \equiv dy/ds \tag{25.6}$$

$$K_x \equiv B'/(B\rho) + \rho^{-2} \ , \quad K_y \equiv -B'/(B\rho) \tag{25.7}$$

$$B' \equiv \partial B_y/\partial x \ . \tag{25.8}$$

The independent variable s is path length along the design trajectory. This motion is called a *betatron* oscillation because it was initially studied in the context of that type of accelerator. The functions K_x and K_y reflect the transverse focussing—primarily due to quadrupole fields except for the radius of curvature, ρ, term in K_x for a synchrotron—so each equation of motion resembles that for a harmonic oscillator but with spring constants that are a function of position. No terms relating to synchrotron oscillations appear, because their time scale is much longer and in this approximation play no role.

These equations have the form of Hill's equation and so the solution in one plane may be written as

$$x(s) = A\sqrt{\beta(s)}\ \cos(\psi(s) + \delta), \tag{25.9}$$

where A and δ are constants of integration and the phase advances according to $d\psi/ds = 1/\beta$. The dimension of A is the square root of length, reflecting the fact that the oscillation amplitude is modulated by the square root of the amplitude function. In addition to describing the envelope of the oscillation, β also plays the role of an 'instantaneous' $\lambda\!\!\!^{-}$. The wavelength of a betatron oscillation may be some tens of meters, and so typically values of the amplitude function are of the order of meters rather than on the order of the beam size. The beam optics arrangement generally has some periodicity and the amplitude function is chosen to reflect that periodicity. As noted above, a small value of the amplitude function is desired at the interaction point, and so the focussing optics is tailored in its neighborhood to provide a suitable β^*.

The number of betatron oscillations per turn in a synchrotron is called the *tune* and is given by

$$\nu = \frac{1}{2\pi} \oint \frac{ds}{\beta} \ . \tag{25.10}$$

Expressing the integration constant A in the solution above in terms of x, x' yields the *Courant-Snyder invariant*

$$A^2 = \gamma(s)\, x(s)^2 + 2\alpha(s)\, x(s)\, x'(s) + \beta(s)\, x'(s)^2$$

where

$$\alpha \equiv -\, \beta'/2, \quad \gamma \equiv \frac{1+\alpha^2}{\beta} \ . \tag{25.11}$$

(The Courant-Snyder parameters α, β and γ employ three Greek letters which have other meanings and the significance at hand must often be recognized from context.) Because β is a function of position in the focussing structure, this ellipse changes orientation and aspect ratio from location to location but the area πA^2 remains the same.

As noted above the transverse emittance is a measure of the area in x, x' (or y, y') phase space occupied by an ensemble of particles. The definition used in Eq. (25.3) is the area that encloses 39% of a Gaussian beam.

For electron synchrotrons the equilibrium emittance results from the balance between synchrotron radiation damping and excitation from quantum fluctuations in the radiation rate. The equilibrium is reached in a time small compared with the storage time.

For present-day hadron synchrotrons, synchrotron radiation does not play a similar role in determining the transverse emittance. Rather the emittance during storage reflects the source properties and the abuse suffered by the particles throughout acceleration and storage. Nevertheless it is useful to argue as follows: Though x' and x can serve as canonically conjugate variables at constant energy this definition of the emittance would not be an adiabatic invariant when the energy changes during the acceleration cycle. However, $\gamma(v/c)x'$, where here γ is the Lorentz factor, is proportional to the transverse momentum and so qualifies as a variable conjugate to x. So often one sees a normalized emittance defined according to

$$\epsilon_N = \gamma \, \frac{v}{c} \, \epsilon, \tag{25.12}$$

which is an approximate adiabatic invariant, e.g. during acceleration.

25.2.2. ***Phase stability***: The particles in a circular collider also undergo synchrotron oscillations. This is usually referred to as motion in the *longitudinal* degree-of-freedom because particles arrive at a particular position along the accelerator earlier or later than an ideal reference particle. This circumstance results in a finite bunch length, which is related to an energy spread.

For dynamical variables in longitudinal phase space, let us take ΔE and Δt, where these are the energy and time differences from that of the ideal particle. A positive Δt means a particle is behind the ideal particle. The equation of motion is the same as that for a physical pendulum and therefore is nonlinear. But for small oscillations, it reduces to a simple harmonic oscillator:

$$\frac{d^2\Delta t}{dn^2} = -(2\pi\nu_s)^2\Delta t \tag{25.13}$$

where the independent variable n is the turn number and ν_s is the number of synchrotron oscillations per turn, analogous to the betatron oscillation tune defined earlier. Implicit in this equation is the approximation that n is a continous variable. This approximation is valid provided $\nu_s \ll 1$, which is usually well satisfied in practice.

In the high-energy limit, where $v/c \approx 1$,

$$\nu_s = \left[\frac{h\eta\, eV\ \cos\phi_s}{2\pi E}\right]^{1/2} \quad . \tag{25.14}$$

There are four as yet undefined quantities in this expression: the harmonic number h, the slip factor η, the maximum energy eV gain per turn from the acceleration system, and the synchronous phase ϕ_s. The frequency of the RF system is normally a relatively high multiple, h, of the orbit frequency. The slip factor relates the fractional change in the orbit period τ to changes in energy according to

$$\frac{\Delta\tau}{\tau} = \eta\frac{\Delta E}{E} \quad . \tag{25.15}$$

At sufficiently high energy, the slip factor just reflects the relationship between path length and energy, since the speed is a constant; η is positive for all the synchrotrons in the "Tables of Collider Parameters" (Sec. 26).

The synchronous phase is a measure of how far up on the RF wave the average particle must ride in order to maintain constant energy to counteract of synchrotron radiation. That is, $\sin\phi_s$ is the ratio of the energy loss per turn to the maximum energy per turn that can be provided by the acceleration system. For hadron colliders built to date, $\sin\phi_s$ is effectively zero. This is not the case for electron storage rings; for example, the electron ring of HERA runs at a synchronous phase of 45°.

Now if one has a synchrotron oscillation with amplitudes $\widehat{\Delta t}$ and $\widehat{\Delta E}$,

$$\Delta t = \widehat{\Delta t}\ \sin(2\pi\nu_s n)\ , \qquad \Delta E = \widehat{\Delta E}\ \cos(2\pi\nu_s n) \tag{25.16}$$

then the amplitudes are related according to

$$\widehat{\Delta E} = \frac{2\pi\nu_s E}{\eta\tau}\widehat{\Delta t} \quad . \tag{25.17}$$

The longitudinal emittance ϵ_ℓ may be defined as the phase space area bounded by particles with amplitudes $\widehat{\Delta t}$ and $\widehat{\Delta E}$. In general, the longitudinal emittance for a given amplitude is found by numerical integration. For $\sin\phi_s = 0$, an analytical expression is as follows:

$$\epsilon_\ell = \left[\frac{2\pi^3 EeVh}{\tau^2\eta}\right]^{1/2} (\widehat{\Delta t})^2 \tag{25.18}$$

Again, a Gaussian is a reasonable representation of the longitudinal profile of a well-behaved beam bunch; if $\sigma_{\Delta t}$ is the standard deviation of the time distribution, then the bunch length can be characterized by

$$\ell = c\,\sigma_{\Delta t} \ . \tag{25.19}$$

In the electron case the longitudinal emittance is determined by the synchrotron radiation process just as in the transverse degrees of freedom. For the hadron case the history of acceleration plays a role and because energy and time are conjugate coordinates, the longitudinal emittance is a quasi-invariant.

For HEP bunch length is a significant quantity because if the bunch length becomes larger than β^* the luminosity is adversely affected. This is because β grows parabolically as one proceeds away from the IP and so the beam size increases thus lowering the contribution to the luminosity from such locations. Further discusssion and modified expressions for the luminosity may be found in Furman and Zisman [3].

25.2.3. ***Synchrotron radiation*** [4]: A relativistic particle undergoing centripetal acceleration radiates at a rate given by the Larmor formula multiplied by the 4th power of the Lorentz factor:

$$P = \frac{1}{6\pi\epsilon_0}\frac{e^2a^2}{c^3}\gamma^4 . \tag{25.20}$$

Here, $a = v^2/\rho$ is the centripetal acceleration of a particle with speed v undergoing deflection with radius of curvature ρ. In a synchrotron that has a constant radius of curvature within bending magnets, the energy lost due to synchrotron radiation per turn is the above multiplied by the time spent in bending magnets, $2\pi\rho/v$. Expressed in familiar units, this result may be written

$$W = 8.85\times 10^{-5} E^4/\rho \ \text{ MeV per turn} \tag{25.21}$$

for electrons at sufficiently high energy that $v \approx c$. The energy E is in GeV and ρ is in kilometers. The radiation has a broad energy spectrum which falls off rapidly above the *critical energy*, $E_c = (3c/2\rho)\hbar\gamma^3$. Typically, E_c is in the hard x-ray region.

The characteristic time for synchrotron radiation processes is the time during which the energy must be replenished by the acceleration system. If f_0 is the orbit frequency, then the characteristic time is given by

$$\tau_0 = \frac{E}{f_0 W} \ . \tag{25.22}$$

Oscillations in each of the three degrees of freedom either damp or antidamp depending on the design of the accelerator. For a simple separated-function alternating gradient synchrotron, all three modes damp. The damping time constants are related by Robinson's Theorem, which, expressed in terms of τ_0, is

$$\frac{1}{\tau_x} + \frac{1}{\tau_y} + \frac{1}{\tau_s} = 2\frac{1}{\tau_0} \ . \tag{25.23}$$

Even though all three modes may damp, the emittances do not tend toward zero. Statistical fluctuations in the radiation rate excite synchrotron oscillations and radial betatron oscillations. Thus there is an equilibrium emittance at which the damping and excitation are in balance. The vertical emittance is non-zero due to horizontal-vertical coupling.

Polarization can develop from an initially unpolarized beam as a result of synchrotron radiation. A small fraction $\approx E_c/E$ of the radiated power flips the electron spin. Because the lower energy state is that in which the particle magnetic moment points in the same direction as the magnetic bend field, the transition rate toward this alignment is larger than the rate toward the reverse orientation. An equilibrium polarization of 92% is predicted, and despite a variety of depolarizing processes, polarization above 80% has been observed at a number of facilities.

The radiation rate for protons is of course down by a factor of the fourth power of the mass ratio, and is given by

$$W = 7.8 \times 10^{-3} E^4/\rho \ \ \text{keV per turn} \tag{25.24}$$

where E is now in TeV and ρ in km. For the LHC, synchrotron radiation presents a significant load to the cryogenic system, and impacts magnet design due to gas desorption and secondary electron emission from the wall of the cold beam tube. The critical energy for the LHC is 44 eV.

25.2.4. ***Beam-beam tune shift*** [5]: In a bunch-bunch collision the particles of one bunch see the other bunch as a nonlinear lens. Therefore the focussing properties of the ring are changed in a way that depends on the transverse oscillation amplitude. Hence there is a spread in the frequency of betatron oscillations.

There is an extensive literature on the subject of how large this tune spread can be. In practice, the limiting value is hard to predict. It is consistently larger for electrons because of the beneficial effects of damping from synchrotron radiation.

In order that contributions to the total tune spread arise only at the detector locations, the beams in a multibunch collider are kept apart elsewhere in the collider by a variety of techniques. For equal energy particles of opposite charge circulating in the same vacuum chamber, electrostatic separators may be used assisted by a crossing angle if appropriate. For particles of equal energy and of the same charge, a crossing angle is needed not only for tune spread reasons but to steer the particles into two separate beam pipes. In HERA, because of the large ratio of proton to electron energy, separation can be achieved by bending magnets.

25.2.5. *Luminosity lifetime*: In electron synchrotrons the luminosity degrades during the store primarily due to particles leaving the phase stable region in longitudinal phase space as a result of quantum fluctuations in the radiation rate and bremsstrahlung. For hadron colliders the luminosity deteriorates due to emittance dilution resulting from a variety of processes. In practice, stores are intentionally terminated when the luminosity drops to the point where a refill will improve the integrated luminosity, while synchrotron radiation facilitates a "topping-up" process in electron-positron rings to provide continuous luminosity.

25.2.6. *Luminosity: Achieved and Desired*:

Many years ago, the CERN Intersecting Storage Rings set an enviable luminosity record for proton-proton collisions at 1.3×10^{32} cm^{-2}s^{-1}. Not until the Summer of 2005 was this level matched in a proton-antiproton collider as the Tevatron approached an integrated luminosity of 1 fb^{-1} in its Run II. Thus far, electrons have proved more amenable; KEKB has reached 1.5×10^{34} cm^{-2}s^{-1}, with the implication of integrated luminosity in the 100 fb^{-1}yr^{-1} range.

The luminosity specification for a yet-to-be-built system is a complicated process, involving considerations of organization, funding, politics and so forth outside the scope of this paper. In the electron world, the $1/s$ dependence of cross sections such as "Higgs-strahlung" plays an important role. In the complex hadron environment, the higher luminosity potential of the LHC helped offset the otherwise totally adverse consequences of the SSC project cancellation. As a historical observation, it is interesting that although improvement directions of present and past facilities could not spelled out in advance, such opportunities have proved to be of critical value to the advance of HEP.

25.3. Prospects

While this update is in preparation, it is interesting to recall that exactly 20 years ago, the Tevatron was entering its commissioning phase as a proton-antiproton collider to provide a c.m.s. energy at the 2 TeV level. Now, the next major step in discovery reach is approaching. The LHC is scheduled to begin operation in 2007, with its proton beams colliding at 14 TeV c.m.s., and at a luminosity two orders of magnitude above that reached in the Tevatron. LHC progress may be followed at the website provided by CERN [6].

A concerted international effort is underway aimed toward the construction of the electron-positron counterpart of the LHC – a linear collider to operate at the 0.5–1.0 TeV c.m.s. level. A recent decision of an International Technical Review Panel decided in favor of the superconducting RF approach for the main linacs [7]. A current goal is to develop the design report by the end of 2006.

References:

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