3. BIG-BANG COSMOLOGY

Revised November 1993 by K.A. Olive.

At early times, and today on a sufficently large scale, our Universe is very nearly homogeneous and isotropic. The most general space-time metric for a homogeneous, isotropic space is the Friedmann-Robertson-Walker metric (with $c = 1$) [1,2,3]:

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] . \tag{3.1}$$

$R(t)$ is a scale factor for distances in comoving coordinates. With appropriate rescaling of the corrdinates, κ can be chosen to be $+1$, -1, or 0, corresponding to closed, open, or spatially flat geometries. Einstein's equations lead to the Friedmann equation

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi\, G_N\, \rho}{3} - \frac{\kappa}{R^2} + \frac{\Lambda}{3} , \tag{3.2}$$

as well as to

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G_N}{3}\ (\rho + 3p) , \tag{3.3}$$

where $H(t)$ is the Hubble parameter, ρ is the total mass-energy density, p is the isotropic pressure, and Λ is the cosmological constant. (For limits on Λ, see the Table of Astrophysical Constants; we will assume here $\Lambda = 0$.) The Friedmann equation serves to define the density parameter Ω_0 (subscript 0 indicates present-day values):

$$\kappa/R_0^2 = H_0^2(\Omega_0 - 1) , \qquad \Omega_0 = \rho_0/\rho_c ; \tag{3.4}$$

and the critical density is defined as

$$\rho_c \equiv \frac{3H^2}{8\pi\, G_N} = 1.88 \times 10^{-29}\, h^2 \text{ g cm}^{-3} , \tag{3.5}$$

with

$$H_0 = 100\, h_0 \text{ km s}^{-1} \text{ Mpc}^{-1} = h_0/(9.78 \text{ Gyr}) . \tag{3.6}$$

Observational bounds give $0.4 < h_0 < 1$. The three curvature signatures $\kappa = +1, -1$, and 0 correspond to $\Omega_0 > 1$, < 1, and $= 1$. Knowledge of Ω_0 is even poorer than that of h_0. Luminous matter (stars and associated material) contribute $\Omega_{\text{lum}} \leq 0.01$. There is no lack of evidence for copious amounts of dark matter: rotation curves of spiral galaxies, virial estimates of cluster masses, gravitational lensing by clusters and individual galaxies, and so on. The minimum amount of dark matter required to explain the flat rotation curves of spiral galaxies only amounts to $\Omega_0 \sim 0.1$, while estimates for Ω_0 based upon cluster virial masses suggests $\Omega_0 \sim 0.2 - 0.4$. The highest estimates for the mass density come from studies of the peculiar motions of galaxies (including our own); estimates for Ω_0 obtained by relating peculiar velocity measurements to the distribution galaxies within a few hundred Mpc approach unity. A conservative range for the mass density is: $0.1 \leq \Omega_0 \leq 2$. The excess of Ω_0 over $\Omega_{\rm lum}$ leads to the inference that most of the matter in the Universe is nonluminous dark matter.

In an expanding universe, the wavelength of light emitted from a distant source is shifted towards the red. The redshift z is defined such that $1+z$ is the ratio of the detected wavelength (λ) to emitted (laboratory) wavelength ($\lambda_{\rm e}$) of some electromagnetic spectral feature. It follows from the metric given in Eq. (3.1) that

$$1+z = \lambda/\lambda_{\rm e} = R_0/R_{\rm e} \tag{3.7}$$

where $R_{\rm e}$ is the value of the scale factor at the time the light was emitted. For light emitted in the not too distant past, one can expand $R_{\rm e}$ and write $R_{\rm e} \simeq R_0 + (t_{\rm e} - t_0)\dot{R}_0$. For small (compared to H_0^{-1}) $\Delta t = (t_{\rm e} - t_0)$, Eq. (3.7) takes the form of Hubble's law

$$z \approx \Delta t \frac{\dot{R}_0}{R_0} \approx \ell H_0 \ , \tag{3.8}$$

where ℓ is the distance to the source.

Energy conservation implies that

$$\dot{\rho} = -3(\dot{R}/R)(\rho + p) \ , \tag{3.9}$$

so that for a matter-dominated ($p = 0$) universe $\rho \propto R^{-3}$, while for a radiation-dominated ($p = \rho/3$) universe $\rho \propto R^{-4}$. Thus the less singular curvature term κ/R^2 in the Friedmann equation can be neglected at early times when R is small. If the Universe expands adiabatically, the entropy per comoving volume ($\equiv R^3 s$) is constant, where the entropy density is $s = (\rho + p)/T$ and T is temperature. The energy density of radiation can be expressed (with $\hbar = c = 1$) as

$$\rho_r = \frac{\pi^2}{30} N(T)(kT)^4 \ , \tag{3.10}$$

where $N(T)$ counts the effectively massless degrees of freedom of bosons and fermions:

$$N(T) = \sum_B g_B + \frac{7}{8} \sum_F g_F \ . \tag{3.11}$$

For example, for $m_\mu > kT > m_e$, $N(T) = g_\gamma + 7/8\,(g_e + 3g_\nu) = 2 + 7/8\,[4 + 3(2)] = 43/4$. For $m_\pi > kT > m_\mu$, $N(T) = 57/4$. At temperatures less than about 1 MeV, neutrinos have decoupled from the thermal background, *i.e.*, the weak interaction rates are no longer fast enough compared with the expansion rate to keep neutrinos in equilibrium with the remaining thermal bath consisting of $\gamma, e^\pm$.

Furthermore, at temperatures $kT < m_e$, by entropy conservation, the ratio of the neutrino temperature to the photon temperature is given by $(T_\nu/T_\gamma)^3 = g_\gamma/(g_\gamma + \frac{7}{8}g_e) = 4/11$.

In the early Universe when $\rho \approx \rho_r$, then $\dot{R} \propto 1/R$, so that $R \propto t^{1/2}$ and $Ht \to 1/2$ as $t \to 0$. The time-temperature relationship at very early times can then be found from the above equations:

$$t = \frac{2.42}{\sqrt{N(T)}} \left(\frac{1\ \text{MeV}}{kT}\right)^2 \ \text{sec} \ . \tag{3.12}$$

At later times, since the energy density in radiation falls off as R^{-4} and the energy density in non-relativistic matter falls off as R^{-3}, the Universe eventually became matter dominated. The epoch of matter-radiation density equality is determined by equating the matter density at t_{eq}, $\rho_m = \Omega_0 \rho_c (R_0/R_{\text{eq}})^3$ to the radiation density, $\rho_r = (\pi^2/30)[2 + (21/4)(4/11)^{4/3}](kT_0)^4 (R_0/R_{\text{eq}})^4$ where T_0 is the present temperature of the microwave background (see below). Solving for $(R_0/R_{\text{eq}}) = 1 + z_{\text{eq}}$ gives

$$\begin{aligned}
z_{\text{eq}} + 1 &= \Omega_0 h_0^2/4.2 \times 10^{-5} = 2.4 \times 10^4\, \Omega_0 h_0^2 \ ; \\
kT_{\text{eq}} &= 5.6\, \Omega_0 h_0^2\, \text{eV} \ ; \\
t_{\text{eq}} &\approx 0.39 (\Omega_0 H_0^2)^{-1/2} (1 + z_{\text{eq}})^{-3/2} \\
&= 3.2 \times 10^{10} (\Omega_0 h_0^2)^{-2}\, \text{sec} \ .
\end{aligned} \tag{3.13}$$

Prior to this epoch the density was dominated by radiation (relativistic particles; see Eq. (3.10)), and at later epochs matter density dominated. Atoms formed at $z \approx 1300$, and by $z_{\text{dec}} \approx 1100$ the free electron density was low enough that space became essentially transparent to photons and matter and radiation were decoupled. These are the photons observed in the microwave background today.

The age of the Universe today, t_0, is related to both the Hubble parameter and the value of Ω_0 (still assuming that $\Lambda = 0$). In the Standard Model, $t_0 \gg t_{\text{eq}}$ and we can write

$$t_0 = H_0^{-1} \int_0^1 \left(1 - \Omega_0 + \Omega_0 x^{-1}\right)^{-1/2} dx \ . \tag{3.14}$$

Constraints on t_0 yield constraints on the combination $\Omega_0 h_0^2$. For example, $t_0 \geq 13 \times 10^9$ yr implies that $\Omega_0 h_0^2 \leq 0.25$ for $h_0 \geq 0.5$, or $\Omega_0 h_0^2 \leq 0.45$ for $h_0 \geq 0.4$, while $t_0 \geq 10 \times 10^9$ yr implies that $\Omega_0 h_0^2 \leq 0.8$ for $h_0 \geq 0.5$, or $\Omega_0 h_0^2 \leq 1.1$ for $h_0 \geq 0.4$.

The present temperature of the microwave background is $T_0 = 2.726 \pm 0.005$ K as measured by COBE [4], and the number density of photons $n_\gamma = (2\zeta(3)/\pi^2)(kT_0)^3 \approx 411\ \text{cm}^{-3}$. The energy density in photons (for which $g_\gamma = 2$) is $\rho_\gamma = (\pi^2/15)(kT_0)^4$. At the present epoch, $\rho_\gamma = 4.65 \times 10^{-34}\,\text{g cm}^{-3} = 0.26\ \text{eV cm}^{-3}$. For nonrelativistic matter (such as baryons) today, the energy density is $\rho_B = m_B n_B$ with $n_B \propto R^{-3}$, so that for most of the history of the Universe n_B/s is constant. Today, the entropy density is related to the photon density by $s = (4/3)(\pi^2/30)[2 + (21/4)(4/11)](kT_0)^3 = 7.0\, n_\gamma$. Big Bang nucleosynthesis calculations limit $\eta = n_B/n_\gamma$ to $2.8 \times 10^{-10} \leq \eta \leq 4.0 \times 10^{-10}$. The parameter η is also related to the portion of Ω in baryons

$$\Omega_B = 3.66 \times 10^7 \eta\, h_0^{-2}\, (T_0/2.726\ \mathrm{K})^3\ , \tag{3.15}$$

so that $0.010 < \Omega_B\ h_0^2 < 0.015$, and hence the Universe cannot be closed by baryons.

References:

1. S. Weinberg, *Gravitation and Cosmology*, John Wiley and Sons (1972).
2. G. Börner, *The Early Universe: Facts and Fiction*, Springer-Verlag (1988).
3. E.W. Kolb and M.S. Turner, *The Early Universe*, Addison-Wesley (1990).
4. J.C. Mather *et al.*, Astrophys. J. **420**, 439 (1994). Error quoted here is 1σ.